FIC in a Variational Framework: 
An Application to the Diffusion-Absorption Problem

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### TABLE OF CONTENTS

§1. **INTRODUCTION** ........................................... 1

§2. **MODIFIED VARIATIONAL FORMS** .................. 2

§3. **THE DIFFUSION ABSORPTION PROBLEM** ........ 2
  §3.1. The Model Problem .................................. 3
  §3.2. The Ritz Solution .................................. 4
  §3.3. The FIC Functional ................................ 5

§4. **THE RITZ FIC EQUATIONS** ......................... 6
  §4.1. Element Equations .................................. 6
  §4.2. Finding $\alpha$ by LDE Consistency .............. 6
  §4.3. Finding $\alpha$ by Positivity ...................... 7
  §4.4. Finding $\alpha$ by Exact Nodal Matching ......... 8
  §4.5. What Happens if $\alpha$ is Imaginary? .......... 8

§5. **NUMERICAL EXPERIMENTS** ......................... 9
  §5.1. Results for $w = 1000$ ............................ 9
  §5.2. Results for $w = 25$ ............................... 10
  §5.3. Results for $w = -25$ ............................. 11

§6. **CONCLUSIONS** ........................................ 12

References ................................................. 12
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An Application to the Diffusion-Absorption Problem

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Abstract
This article outlines the use of Finite Calculus (FIC) ideas in a variational framework, using modified
state functions. The technique is applied to the finite element discretization of the steady-state,
one-dimensional, diffusion-absorption equations. Linear shape functions are directly inserted into a
FIC-modified functional. The functional, as well as the Ritz-FIC equations derived thereto, contain
a free stabilization parameter coming from the function modification process. It is shown that the
parameter can be used to produce nodally exact solutions for all values of the absorption/diffusion
coefficient.

Keywords: finite increment calculus, variational principles, Ritz method, functional modification,
stabilization, finite elements, diffusion, absorption, nodally exact solution

§1. INTRODUCTION

The Finite Calculus (FIC) has been developed over the past 5 years [1–10] as a general purpose tool
for improving the stability and accuracy of volume-based discretizations of equations of mathematical
physics. Consider a problem governed by the residual equation

\[ \mathbf{r} (\mathbf{u}) = 0, \]  

where \( \mathbf{u} \) is an array of \( n \) primary variables. These in turn are functions of the independent variables
\( \mathbf{x} \), which may include time. Generally (1) is an ordinary or partial differential equation, to be solved
by numerical methods. Introduce \( n \) characteristic lengths \( h_i \) collected in array \( \mathbf{h} \), where \( h_i \) is paired
with the function \( u_i \). These lengths can be viewed as linked through as yet unspecified means, to
mesh dimensions. Using flux balance arguments [8] one constructs a modified residual

\[ \mathbf{r} (\mathbf{u}) + \mathbf{r}_h (\mathbf{u}, \mathbf{h}) = 0. \]  

The simplest form of \( \mathbf{r}_h \) is \(-\frac{1}{2} \nabla \mathbf{r} \mathbf{h} \), where \( \nabla \mathbf{r} \) is the gradient matrix of \( \mathbf{r} \) with respect to the independent
variables.

The discretization process — usually Galerkin-based FEM — is applied to (2) instead of (1). Consis-
tency with the original equation requires that the augmentation term \( \mathbf{r}_h \to 0 \) as \( h_i \to 0 \). But the
philosophy of FIC, as emphasized in its name, is that in practice the \( h \) remain finite. The key goal is to pick \( r_h \) and \( h \) so that stability and accuracy characteristics of the solution for a given mesh are improved.

FIC has been principally used [1–7] for the solution of fluid mechanics equations involving flow, advection, diffusion, gravity waves and reactions. In that application realm it competes with stabilization schemes such as SUPG, Galerkin Least Squares (GLS) and subgrid scale (SGS) methods.

In a study of FIC methods for solid mechanics [11] it was found that a variational form formally analogous to the Minimum Potential Energy principle could be obtained by modifying the displacement, strain and stress fields in a manner similar to that done for the residual in the foregoing description, and adjusting their variations. The approach technically falls into the class of variational principles with noncommutative variations [12], also called modified variational principles in the literature [13]. That finding provides the departure point for the present study.

§2. MODIFIED VARIATIONAL FORMS

Suppose that (1) is derivable from a functional \( J[u] \) in the sense that \( r(u) = 0 \) are the Euler-Lagrange equations of \( J \). The first variation is

\[
\delta J[u] = \delta u^T r(u) \tag{3}
\]

Introduce a modified primary variable field:

\[
\tilde{u} \overset{\text{def}}{=} u + u_h(h) \tag{4}
\]

such that \( u_h \to 0 \) as \( h \to 0 \). The choice considered here, suggested by the study in [11], is

\[
\tilde{u} = u - \frac{1}{2} h \alpha^T \nabla u. \tag{5}
\]

Here \( h \) is an overall characteristic length, \( \alpha \) collects scaling parameters, and the factor \( -\frac{1}{2} \) is for convenience in matching to the standard FIC method. Introducing (5) into \( J \) yields the modified functional:

\[
\tilde{J}_h = J[\tilde{u}] = J + J_h \tag{6}
\]

where the augmentation term \( J_h \) vanishes as \( h \to 0 \). The Euler-Lagrange equation changes to

\[
\delta \tilde{J}_h[u] = \delta u^T \left( r(u) + \tilde{r}_h(u) \right) \tag{7}
\]

This has formally the same configuration as (3), and shares with it the property that as \( h \to 0 \) the Euler-Lagrange equation reduces to (1). But in general starting with \( r_h(u) \) of FIC, namely that in (2), does not reproduce \( \tilde{r}_h \). To avoid confusion we qualify (7) as the FIC variational residual. The functional \( \tilde{J}_h \) will be called the FIC-modified functional, or FIC functional for brevity. (The superposed tildes are eventually dropped for brevity when there is no danger of confusion.) Summarizing:

 Modify primary fields \[\rightarrow\] Insert in \( J \) \[\rightarrow\] vary \( \tilde{J}_h \) \[\rightarrow\] FIC variational residual \[8\]

The numerical approximation is obtained by working with \( \tilde{J}_h \) in the usual way, assuming that \( h \) is known. The residual may be used to study stability and accuracy properties of the approximation.

The remainder of this article illustrates the procedure for the one-dimensional diffusion-absorption equation. This problem has been recently examined by Oñate, Miquel and Hauke [7] from a FIC-Galerkin standpoint. That study includes advection terms which are not considered here.
§3. THE DIFFUSION ABSORPTION PROBLEM

The governing differential equation that models a one-dimensional, steady state, diffusion-absorption process is

\[ \frac{d}{dx} \left( k \frac{du}{dx} \right) - su + Q = 0, \quad \text{in } x \in [x_0, x_1] \]  

(9)

In this equation \( u \) is the state variable, \( x \in [x_0, x_1] \) is the problem domain, \( k \geq 0 \) is the diffusion, \( s \geq 0 \) is the absorption (also called dissipation or destruction parameter) and \( Q \) the source term. Using primes to denote differentiation with respect to \( x \), the foregoing ODE can be abbreviated to

\[ (k u')' - su + Q = 0. \]  

(10)

With the flux defined as \( q = k(du/dx) = ku' \), the boundary conditions can be stated as

\[ u = \hat{u} \quad \text{on } \Gamma^u, \quad q = \hat{q}, \quad \text{on } \Gamma^q. \]  

(11)

where \( \Gamma^u \) and \( \Gamma^q \) are the Dirichlet and Neumann boundaries, respectively. For the one-dimensional problems these consists of four combinations at the ends of the problem domain.

This problem admits a classical variational formulation. Introduce the functional

\[ J[u] = \int_{x_0}^{x_1} \left( \frac{1}{2} k (u')^2 + \frac{1}{2} su^2 - Qu \right) dx. \]  

(12)

Taking the first variation \( \delta J = 0 \) over admissible functions \( u(x) \) that satisfy the essential BCs yields the differential equation (10) as Euler-Lagrange equation, and the flux BCs in (11) as natural boundary condition.

§3.1. The Model Problem

If \( k > 0 \), a model form of (10) is obtained by introducing the dimensionless coefficient

\[ w = \frac{sa^2}{k}, \]  

(13)

where \( a = x_1 - x_0 \) is the length of the problem domain. This coefficient characterizes the relative importance of absorption over diffusion. The problem domain is adjusted to extend from \( x_0 = -\frac{1}{2}a \) to \( x_1 = \frac{1}{2}a \) for convenience. We will assume zero source: \( Q = 0 \), and Dirichlet boundary conditions at both ends: \( u(-\frac{1}{2}a) = u_m \) and \( u(\frac{1}{2}a) = u_p \). We can therefore state the model problem as

\[ u'' - \frac{w}{a^2} u = 0 \quad \text{for } x \in [-\frac{1}{2}a, \frac{1}{2}a], \quad u(-\frac{1}{2}a) = u_m, \quad u(\frac{1}{2}a) = u_p. \]  

(14)

The associated functional is

\[ J[u] = \int_{-a}^{a} \left( (u')^2 + \frac{1}{2} \frac{w}{a^2} u^2 \right) dx. \]  

(15)
where variation is taken over continuous \( u(x) \) that satisfy the Dirichlet BCs.

If \( w \neq 0 \) the exact solution of the model problem (14) is

\[
    u(x) = \frac{\sinh(\sqrt{w}(1 - 2x/a)) u_m + \sinh(\sqrt{w}(1 + 2x/a)) u_p}{\sinh(2\sqrt{w})}.
\]  

This form becomes 0/0 for \( w = 0 \) and suffers from cancellation errors if \(|w|\) is very small, say \(|w| < 10^{-6}\). For that case a Taylor series about \( w = 0 \) gives, to first order in \( w \):

\[
    u(x) \approx u_m((12a^3 - 24a^2x) + w(-3a^3 + 2a^2x + 12ax^2 - 8x^3))/(24a^3) + u_p((12a^3 + 24a^2x) + w(-3a^3 - 2a^2x + 12ax^2 + 8x^3))/(24a^3).
\]  

The exact solution is displayed in Figure 1 for \( w = 1000, 100, 0, -25, u_m = 1 \) and \( u_p = 3 \). If \( w = 0 \) the solution is a straight line. As \( w \) grows, exponential-growth boundary layers appear at Dirichlet boundaries. This is illustrated by the upper plots in Figure 1. If \( w = 1000 \) the solution is virtually zero over most of the problem domain, with two sharp boundary layers.

If \( w < 0 \) the solution (16) involves complex exponentials and the response is oscillatory. This case has no physical significance for modeling diffusion-absorption, but a similar equation is applicable to several problems in solid mechanics.

§3.2. The Ritz Solution

A standard FEM solution is easily constructed by the variational formulation. Divide the domain into \( N^e \) two-node elements of length \( L^e \). The end nodes are \( i \) and \( j \), with coordinates \( x_i \) and \( x_j \), and node values \( u_i \) and \( u_j \), respectively. Assume the piecewise linear interpolation

\[
    u(x) = u_i N_i(x) + u_j N_j(x)
\]
where \( N_i(x) = (x - x_i)/L_e \), \( N_j = (x_j - x)/L_e \) and \( L_e = x_j - x_i \) are the well known linear shape functions. Inserting into (15) gives, for the model problem, the element stiffness equations

\[
K_e u_e = \frac{1}{L_e} \left[ \begin{array}{cc} 1 + \frac{1}{3} \chi^2 w & -1 + \frac{1}{6} \chi^2 w \\ -1 + \frac{1}{6} \chi^2 w & 1 + \frac{1}{3} \chi^2 w \end{array} \right] \left[ \begin{array}{c} u_i \\ u_j \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \end{array} \right], \quad \chi = L_e/a. \quad (19)
\]

If \( w = 0 \) this element relation gives, upon assembly, the linear response correctly. However if \( w \neq 0 \), the use of (19), a scheme that may be labeled “unstabilized Ritz,” displays serious drawbacks:

(a) If \( w \) is very large the solution oscillates over coarse meshes. See Figure 2 for 8 elements and \( w = 1000 \). Negative \( u \) values are physically incorrect, which renders the solution useless.

(b) For moderate \( w \) the accuracy is poor.

These well known shortcomings are usually handled by Galerkin stabilization schemes, for example with suitable weight functions. But this amounts to using a “mirror equation” approach to what is essentially a self-adjoint problem.

§3.3. The FIC Functional

We try to stay within the Ritz method and piecewise linear shape functions, but change the functional by the method outlined in §2. For this problem, the FIC function modification technique consists of formally replacing

\[
\tilde{u}(x) = u(x) - \frac{1}{2} hu'(x), \quad \tilde{u}'(x) = u'(x) - \frac{1}{2} hu''(x). \quad (20)
\]

These \( \tilde{u}(x) \) and \( \tilde{u}'(x) \) are formally inserted into (15). The tildes are then suppressed for brevity. This scheme yields a modified functional \( J_h[u] \), where \( h \) is the FIC steplength. That \( h \) was derived in the original FIC by flux balancing arguments [8]. In the present case \( h \) may be simply viewed as a free parameter with dimension of length.

For piecewise linear shape functions \( u''(x) \) vanishes over each element, and the second replacement in (20) may be skipped. With this simplification the modified functional is

\[
J_h[u] = \int_{-a}^{a} \left( (u')^2 + \frac{1}{2} \frac{w}{a^2} (u - \frac{1}{2} hu'(x))^2 \right) dx. \quad (21)
\]
The Euler-Lagrange equation given by $\delta J_h[u] = 0$ is

$$(1 + \frac{wh^2}{4a^2})u'' - \frac{w}{a^2}u = 0.$$  \hspace{1cm} (22)$$

From this the FIC variational residual follows as $\delta u \left[(1 + \frac{1}{4}wh^2/a^2)u'' - (w/a^2)u\right]$. The expression (22) shows that a nonzero $h$ injects artificial diffusion if $w > 0$. Furthermore, the sign of $h$ makes no difference in the interior of the problem domain. As $h \to 0$ the original ODE (14) is recovered. But the key idea is to keep $h$ finite and directly related to mesh size.

§4. THE RITZ FIC EQUATIONS

We study here the use of the FIC functional (21) in conjunction with the piecewise-linear shape functions (18) to construct Ritz finite-element equations for the model diffusion-absorption problem. The steplength $h = h^e$ may in fact change from element to element. For convenience define $h^e = \alpha^e L^e$ where $\alpha^e$ is a dimensionless parameter to be determined. The following analysis is restricted to equal size elements and the same $\alpha$ for all elements.

§4.1. Element Equations

The resulting element equations, called the Ritz FIC equations in the sequel, are

$$\frac{1}{L^e} \begin{bmatrix} 1 + (\frac{1}{3} + \frac{1}{2}\alpha + \frac{1}{4}\alpha^2) \chi^2 w & -1 + (\frac{1}{6} - \frac{1}{4}\alpha^2) \chi^2 w \\ -1 + (\frac{1}{6} - \frac{1}{4}\alpha^2) \chi^2 w & 1 + (\frac{1}{3} - \frac{1}{2}\alpha + \frac{1}{4}\alpha^2) \chi^2 w \end{bmatrix} \begin{bmatrix} u_i \\ u_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \chi = L^e/a.$$ \hspace{1cm} (23)$$

The stiffness equations for a patch of two equal-size elements comprising nodes $i, j, k$, as shown in Figure 3, are

$$\frac{1}{L^e} \begin{bmatrix} 1 + (\frac{1}{3} + \frac{1}{2}\alpha + \frac{1}{4}\alpha^2) \chi^2 w & -1 + (\frac{1}{6} - \frac{1}{4}\alpha^2) \chi^2 w & 0 \\ -1 + (\frac{1}{6} - \frac{1}{4}\alpha^2) \chi^2 w & 2 + (\frac{2}{3} + \frac{1}{2}\alpha^2) \chi^2 w & -1 + (\frac{1}{6} - \frac{1}{4}\alpha^2) \chi^2 w \\ 0 & -1 + (\frac{1}{6} - \frac{1}{4}\alpha^2) \chi^2 w & 1 + (\frac{1}{3} - \frac{1}{2}\alpha + \frac{1}{4}\alpha^2) \chi^2 w \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ \hspace{1cm} (24)$$

The remaining issue is to find the values of $\alpha$ to be inserted into the discrete system. Three methods are studied below. Of these the consistency and positivity methods are readily extended to 2D and 3D, although intensive algebraic manipulations may be required. The nodal matching method, which relies on knowledge of the exact solution, appears restricted to one dimension if general geometries and boundary conditions are considered.

Note that the three methods discussed below actually find $\alpha^2$ and not $\alpha$. Which sign of the square root is taken makes no difference for the model problem.
§4.2. Finding \( \alpha \) by LDE Consistency

This technique follows Park and Flaggs [14,15]. Expand \( u(x) \) by Taylor series at the center node \( j \) of the two-element patch of Figure 3, and evaluate at the end nodes:

\[
\begin{align*}
  u_i &= u_j - L^e u'_j + \frac{1}{2} (L^e)^2 u''_j - \frac{1}{6} (L^e)^3 u'''_j + \ldots, \\
  u_k &= u_j + L^e u'_j + \frac{1}{2} (L^e)^2 u''_j + \frac{1}{6} (L^e)^3 u'''_j + \ldots,
\end{align*}
\]  

(25)

These are Laplace transformed — formally on replacing each derivative operator \( (d/dx) \) by the Laplace transform variable \( p \) — and inserted into (22). Solve for the transformed interior value \( \bar{u}_j \) and backtransform to get

\[
\begin{align*}
  w a^2 u_j - u''_j \left( 1 + \frac{(3\alpha^2 - 2)\chi^2}{12} \right) - u^iv^i a^2 \chi^2 \left( 12 - (3\alpha^2 - 2)\chi^2 w \right) + \ldots &= 0.
\end{align*}
\]  

(26)

This is the Limit Differential Equation (LDE) satisfied by the Ritz-FIC Equations as the element size \( L^e \) tends to zero. If we insist on strong consistency with the original ODE (14) the coefficient of the second term must be unity, whence

\[
\alpha^2_C = \frac{2}{3}.
\]  

(27)

Reverting signs, the LDE then becomes \( u''_j - (w/a^2)u_j - \frac{1}{12} u^iv^i a^2 \chi^2 + \ldots = 0 \). The fourth order and higher terms vanish as \( \chi = (L^e/a) \to 0 \).

The value (27) will be called the “consistent \( \alpha \)” because it is obtained by ODE consistency arguments. It is noted that use of this value overestimates the diffusion for all positive \( w \). Consequently it is safe in the sense of providing physically correct solutions. But these can be highly inaccurate for small or moderate \( w \). The examples of §5 illustrate this fact. However this method gives a key result: if \( w \to +\infty, \alpha^2 \to 2/3 \) for consistency. The next two formulas do satisfy that condition.

§4.3. Finding \( \alpha \) by Positivity

Consider the matrix equations (24) of the 2-element patch. Suppose \( u_i > 0 \) and \( u_k > 0 \) are prescribed. Solving for \( u_j \) from the second equation gives

\[
\begin{align*}
  u_j &= \frac{1 - (\frac{1}{6} - \frac{1}{4}\alpha^2)\chi^2 w}{2 + (\frac{2}{3} + \frac{1}{2}\alpha^2)\chi^2 w} (u_i + u_k)
\end{align*}
\]  

(28)

The denominator is positive for any \( \{ \alpha, \chi \} \) if \( w > 0 \). It follows that the condition for \( u_j \geq 0 \) is \( 1 - (\frac{1}{6} - \frac{1}{4}\alpha^2)\chi^2 w \geq 0 \), whence \( \alpha^2 \leq (2/3) - 4/(\chi^2 w) = \alpha^2_P. \) (Here the \( P \) subscript stands for “positivity.”) Thus

\[
\begin{align*}
  \alpha^2_P &= \frac{2}{3} - \frac{4}{\chi^2 w}.
\end{align*}
\]  

(29)

This relation gives a useful bound: if \( \chi^2 w \leq 6 \), \( \alpha \) may be set to zero without impairing positivity. For example, if \( w = 600 \), a mesh of 10 elements (or more) can be used with \( \alpha = 0 \), since \( \chi = 1/10 \) and \( \chi^2 w = 6 \). As discussed in the next paragraph, this does not imply an accurate solution; quite the contrary.
Inserting this $\alpha_p^2$ into the element matrix $K^e$ of (23) cancels out the off-diagonal terms. The assembled $K$ is therefore diagonal. The solution for zero source and Dirichlet conditions at both ends is therefore zero at all interior nodes. This mimics well the physical behavior for large and positive $w$; say $w > 1000$. For positive but smaller $w$ this solution can be way off, but it shows that (29) may be viewed as a lower bound on acceptable values of $\alpha^2$, whereas $\alpha_C^2 = 2/3$ is an upper bound. The numerical results discussed in §5, however, show that these bounds are of little practical value for moderate values of $w$.

§4.4. Finding $\alpha$ by Exact Nodal Matching

Suppose the end node values $u_i$ and $u_k$ of the 2-element patch are set to be those of the exact solution (16). Get the interior node value $u_j$ from the second row of (24). Require matching with the exact solution at that position and solve for $\alpha$. The result can be presented as

$$\alpha_M^2 = \alpha_P^2 + \frac{4e^{\sqrt{w/2}}}{(e^{\sqrt{w/2}} - 1)^2} = \alpha_P^2 + \frac{1}{\sinh^2(\frac{1}{2}\sqrt{w/2})}.$$  \hfill (30)

where $\alpha_P^2$ is given by (29). Obviously $\alpha_M^2 > \alpha_P^2$ as long as $w > 0$. Thus $\alpha_M^2$ also satisfies positivity.

What happens for more than two elements? Given an arbitrary $\chi = L^e/a$, Mathematica was able to find the following generalization:

$$\alpha_M^2 = \alpha_P^2 + \frac{4e^{2\chi\sqrt{w/2}}}{(e^{2\chi\sqrt{w/2}} - 1)^2}. \hfill (31)$$

This recovers (30) if $\chi = \frac{1}{2}$, as can be expected. Use of this $\alpha$ furnishes a nodally exact solution under the following conditions: elements of equal length, constant $w$, zero source and Dirichlet BCs. On the other hand, it is valid for any nonzero $w$, including negative values. Although this appears at first sight surprising, it has been verified by numerical experiments. See the example in §5.3.

The Taylor series of $\alpha_M^2$ as $\chi = L^e/a \rightarrow 0$ (i.e., as the mesh is refined with more elements) is

$$\alpha_M^2 = \frac{1}{3} - \frac{2}{\chi^2w} + \frac{\chi^2w}{30} + O(\chi^4) \hfill (32)$$

This shows that if $w > 0$ eventually $\alpha_M^2$ becomes negative as more elements are added. The opposite conclusion is valid if $w < 0$.

§4.5. What Happens if $\alpha$ is Imaginary?

If $\chi^2w < 6$, $\alpha_P^2 < 0$. Therefore $\alpha_M^2$ may also be negative; note the series (32). If so $\alpha$ is imaginary. The result is the appearance of complex numbers in $K$. If the mesh consists of equal elements, and $w$ is constant, complex numbers occur only in the first and last rows of $K$, and disappear altogether on applying Dirichlet BCs. Aside from this case the use of complex arithmetic is necessary, although the equations remain symmetric.

If avoiding complex arithmetic is deemed desirable, a negative value of $\alpha^2$ should be replaced by zero.
Figure 4. Ritz-FIC results of an 8-element discretization of the diffusion-absorption model problem with \( w = 1000 \), Dirichlet BCs \( u(-\frac{1}{2}) = 1 \) and \( u(\frac{1}{2}) = 3 \), for three values of \( \alpha \), compared to the exact solution.

Table 1. Ritz-FIC 8-element solutions, \( w = 1000 \).

<table>
<thead>
<tr>
<th>Node</th>
<th>( \alpha_C^2 = 0.666667 )</th>
<th>( \alpha_P^2 = 0.410667 )</th>
<th>( \alpha_M^2 = 0.425716 )</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.</td>
<td>1.</td>
<td>1.</td>
<td>1.</td>
</tr>
<tr>
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</tbody>
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§5. NUMERICAL EXPERIMENTS

This section presents numerical results obtained with the Ritz-FIC method for the model diffusion-absorption equation. The problem domain is taken to have unit length \( (a = 1) \) and extends from \( x_0 = -\frac{1}{2}a = -\frac{1}{2} \) through \( x_1 = \frac{1}{2}a = \frac{1}{2} \). The boundary conditions are of Dirichlet type: \( u(-\frac{1}{2}) = 1 \) and \( u(\frac{1}{2}) = 3 \). The domain is divided into 8 elements of equal size; thus \( \chi = L/a = \frac{1}{8} \). Three values of \( w \), ranging from high through negative absorption, are tested.

§5.1. Results for \( w = 1000 \)

The case \( w = 1000 \) exhibits two sharp boundary layers. Over the propagation region, which extends roughly over the middle six elements, \( u(x) \) takes very small positive values, of order down to \( 10^{-10} \). The problem is solved using three \( \alpha \) values: \( \alpha_C^2 = 2/3 \), \( \alpha_P^2 = 2/3 - 4/(\chi^2w) = 101/375 = 0.4106666666666 \) and \( \alpha_M^2 = \alpha_P^2 + e^{5\sqrt{3}/2}/(e^{5\sqrt{3}/2} - 1)^2 = 0.42571643297230183 \). The numerical results are shown in Figure 4 and listed in Table 1, along with the exact solution. As expected the solution for \( \alpha_M^2 \) is nodally exact. On the plot of Figure 4 this solution and the zero-at-all-interior-nodes
solution provided by $\alpha_P^2$ cannot be distinguished within plot resolution, but the relative differences are clear from the listing in Table 1.

For the case $\alpha = 0$, which gives a physically inadmissible solution, see Figure 2.

§5.2. Results for $w = 25$

This case $w = 25$ pertains to moderate absorption. The boundary layers are more diffuse and the variation of the exact solution resembles a parabolic shape. The problem is again solved using three $\alpha$ values: $\alpha_C^2 = 2/3$, $\alpha_P^2 = 2/3 - 4/((\chi^2 w)) = -718/75 = -9.573333333$ and $\alpha_M^2 = \alpha_P^2 + e^{5/(4\sqrt{2})} / (e^{5/(4\sqrt{2})} - 1)^2 = -4.77403874047153$. Observe that for the last two values $\alpha$ is imaginary and complex numbers appear in the first and last row of $K$; however the solution $u(x)$ for Dirichlet boundary conditions is real.

The numerical results are shown in Figure 5 and listed in Table 2, along with the exact solution. Again the solution for $\alpha_M^2$ is nodally exact. The solution for the other two values of $\alpha$ are upper and lower bounds to the exact solution, but remain far from it at all interior nodes.
The results for $\alpha = 0$ differ little from those of $\alpha^2 = 2/3$ and are not shown.

§5.3. Results for $w = -25$

This case $w = -25$ pertains to negative absorption. (As previously observed, this is not physically relevant for fluids, but it holds for some problems in solid mechanics.) Negatives values of $u(x)$ are now physically admissible. The exact solution does not display boundary layers and is in fact oscillatory. The problem is solved using three $\alpha$ values: $\alpha^2_C = 2/3$, $\alpha^2_P = 2/3 - 4/(\chi^2 w) = 818/75 = 10.9066666667$ and $\alpha^2_M = \alpha^2_P + e^{5\sqrt{-1}/(4\sqrt{2})}/(e^{5\sqrt{-1}/(4\sqrt{2})} - 1)^2 = 5.439897503526679$. Note that for the last two values $\alpha^2$ is real and positive. Therefore the assembled matrix $K$ contains only real entries.

The numerical results are shown in Figure 6 and listed in Table 3, along with the exact solution. Again the solution for $\alpha^2_M$ is nodally exact. The computed solutions for the other two values of $\alpha$ are useless.

The results for $\alpha = 0$ differ little from those of $\alpha^2 = 2/3$ and are not shown.
The FIC approach to functional modification permits effective stabilization of the diffusion-absorption problem while staying within the ordinary Ritz-FEM framework. For the model one-dimensional problem it is possible to find a value of the stabilization parameter that is nodally exact for all values of the absorption/diffusion coefficient, including negative ones.

One key difference with the Galerkin approach studied in [7] is that the FIC steplength $h$ — or equivalently its dimensionless counterpart $\alpha$ — enters only linearly in the discrete equations, since it is not included in the weight function. In the Ritz FIC approach, the squared steplength appears automatically. A similar thing would happen in a least squares formulation. This brings up the question of which sign of the square root to take. In the model problem studied here, the sign has no effect on the results.

References


