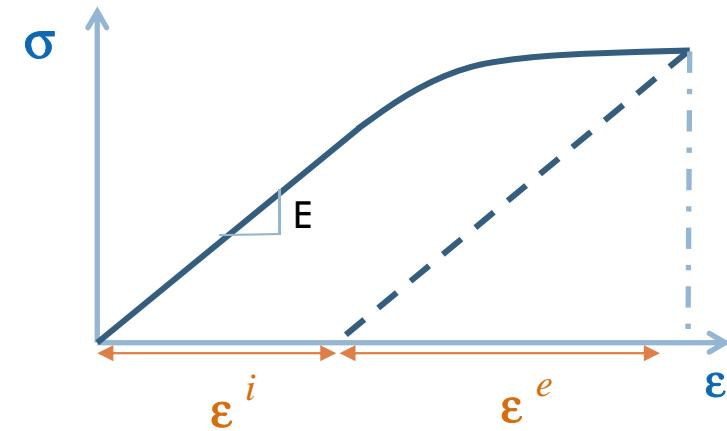


Inelastic models based on additive strain decomposition (plastic material)

- Total strain is decomposed into elastic and inelastic strain counterparts:

$$\varepsilon = \underbrace{\varepsilon^e}_{\text{Elastic strain}} + \underbrace{\varepsilon^i}_{\text{Inelastic strain}}$$

Hypothesis: infinitesimal strains



- Isothermal case:

$$\Delta \theta = 0 \quad \text{No thermal influence}$$

- Variables definition

□ Free variables: $\mathbb{F} := \{\varepsilon\}$

□ Internal variables: $\mathbb{I} := \{\varepsilon^i, \alpha\}$ and evolution equations

□ Dependent variables

$$\begin{cases} \dot{\alpha} = \dot{\alpha}(\varepsilon, \dot{\varepsilon}, \alpha) \\ \dot{\varepsilon}^i = \dot{\varepsilon}^i(\varepsilon, \dot{\varepsilon}, \alpha) \end{cases}$$

$$\mathbb{D} := \{\sigma, \psi\} \rightarrow \begin{cases} \dot{\sigma} = \frac{\partial \sigma(\varepsilon, \varepsilon^i, \alpha)}{\partial \varepsilon} : \dot{\varepsilon} + \frac{\partial \sigma(\varepsilon, \varepsilon^i, \alpha)}{\partial \varepsilon^i} : \dot{\varepsilon}^i + \frac{\partial \sigma(\varepsilon, \varepsilon^i, \alpha)}{\partial \alpha} \dot{\alpha} \\ \dot{\psi} = \frac{\partial \psi(\varepsilon, \varepsilon^i, \alpha)}{\partial \varepsilon} : \dot{\varepsilon} + \frac{\partial \psi(\varepsilon, \varepsilon^i, \alpha)}{\partial \varepsilon^i} : \dot{\varepsilon}^i + \frac{\partial \psi(\varepsilon, \varepsilon^i, \alpha)}{\partial \alpha} \dot{\alpha} \end{cases}$$

Additive strain decomposition (elasto-plastic material)

- Helmholtz free energy:

$$\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) = \underbrace{\frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i)^e : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i)^e}_{\text{Elastic potential}} + \underbrace{\mathcal{H}(\alpha)}_{\text{Hardening potential}} \quad \forall \dot{\boldsymbol{\varepsilon}}$$

- Dissipation: $\mathcal{D} = -\rho_0 (\dot{\psi} + \sigma : \dot{\boldsymbol{\theta}}) + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} \geq 0$

Replacing the free energy, in dissipation yields:

$$\mathcal{D} = \underbrace{(\boldsymbol{\sigma} - \frac{\partial(\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha))}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}})}_{f(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)} + \underbrace{\frac{\partial(\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha))}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}^i}_{g_1(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)} - \underbrace{\frac{\partial(\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha))}{\partial \alpha} \dot{\alpha}}_{g_2(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

REMARK

The following expressions are used:

$$\frac{\partial(\rho_0 \psi)}{\partial \boldsymbol{\varepsilon}} = \frac{\partial(\rho_0 \psi)}{\partial \boldsymbol{\varepsilon}^e} \cdot \underbrace{\frac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\varepsilon}}}_{=1} \quad \text{and} \quad \frac{\partial(\rho_0 \psi)}{\partial \boldsymbol{\varepsilon}^i} = \frac{\partial(\rho_0 \psi)}{\partial \boldsymbol{\varepsilon}^e} \cdot \underbrace{\frac{\partial \boldsymbol{\varepsilon}^e}{\partial \boldsymbol{\varepsilon}^i}}_{-1}$$

$$\mathcal{D} = f(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) : \dot{\boldsymbol{\varepsilon}} + g(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

Coleman's Method

- **Theorem:**

$$\mathcal{D} := f(x, y)\dot{x} + g(x, y) \geq 0 \quad \forall \dot{x} \quad \rightarrow \quad \begin{cases} f(x, y) = 0 \\ g(x, y) \geq 0 \end{cases}$$

- **Proof:**

- If $f(x, y) < 0$ taking $\dot{x} > 0$

such that $f(x, y)\dot{x} + g(x, y) \leq 0 \quad \rightarrow \quad \mathcal{D} = f(x, y)\dot{x} \leq 0$

- If $f(x, y) > 0$ taking $\dot{x} < 0$

such that $f(x, y)\dot{x} + g(x, y) \leq 0 \quad \rightarrow \quad \mathcal{D} = f(x, y)\dot{x} \leq 0$

- If $f(x, y) = 0$ and $\mathcal{D} \geq 0 \quad \forall \dot{x} \quad \rightarrow \quad g(x, y) \geq 0$

NOT POSSIBLE

$$f(x, y) = 0$$

Additive strain decomposition (elasto-plastic material)

- Using the Coleman's theorem for the plastic material model:

$$f(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) = \sigma - \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} = 0 \quad \rightarrow \quad \sigma = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}}$$

$$\mathcal{D} = g(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) = \underbrace{\frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}^i}_{\frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} = \sigma} - \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \alpha} : \dot{\alpha} \geq 0$$

and differentiating the Helmholtz free energy

$$\rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha) = \frac{1}{2} \underbrace{\boldsymbol{\varepsilon}^e : \mathbb{C} : \boldsymbol{\varepsilon}^e}_{(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i)} + \mathcal{H}(\alpha)$$

Constitutive equation

$$\sigma = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \underbrace{\boldsymbol{\varepsilon}^e}_{(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i)}$$

Dissipation \mathcal{D}

$$\mathcal{D} = \sigma : \dot{\boldsymbol{\varepsilon}}^i - \underbrace{\left(\frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \alpha} \right)}_{\mathcal{H}'(\alpha) := -q(\alpha)} : \dot{\alpha} = \sigma : \dot{\boldsymbol{\varepsilon}}^i + \underbrace{q(\alpha)}_{\text{Hardening variable}} \dot{\alpha} \geq 0$$

Strain driven models

- Variables sets:

- The strain is the free variable $\mathcal{F} := \{\boldsymbol{\varepsilon}\}$
- The internal variables are given by: $\mathcal{I} := \{\boldsymbol{\varepsilon}^i, \alpha\}$
- The stresses are dependent variables $\mathcal{D} := \{\boldsymbol{\sigma}, \psi\}$

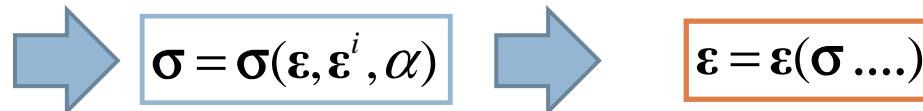
- Potential

- The free energy is required: $\rho_0 \psi(\boldsymbol{\varepsilon}, \theta) = \frac{1}{2} \boldsymbol{\varepsilon}^e : \mathbb{C} : \boldsymbol{\varepsilon}^e + \mathcal{H}(\alpha)$

Coleman's theorem

$$\boldsymbol{\sigma} = \rho_0 \frac{\partial \psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^i, \alpha)}{\partial \boldsymbol{\varepsilon}^e} = \mathbb{C} : (\underbrace{\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^i}_{\boldsymbol{\varepsilon}^e})$$

Constitutive equations



Is it possible to get the inverse
constitutive equation?

Stress driven models

□ Variables sets:

- The strain is the free variable $\mathcal{F} := \{\boldsymbol{\sigma}\}$
- The internal variables are given by: $\mathcal{I} := \{q\} \rightarrow \dot{q} = \zeta(\boldsymbol{\sigma}, q)$ Evolution equation
- The stresses are dependent variables $\mathcal{D} := \{\boldsymbol{\varepsilon}, G\}$

□ Potential

- The Gibbs potential is required:

and its time derivative equation is:

Substituting it in the dissipation yields:

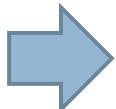
Legendre transform: $\boldsymbol{\varepsilon}, \psi(\boldsymbol{\varepsilon}) \rightarrow \boldsymbol{\sigma}, G(\boldsymbol{\sigma})$

$$\rho_0 G(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \rho_0 \psi(\boldsymbol{\varepsilon}) ; \quad \boldsymbol{\sigma} = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}}$$

$$\rho_0 \dot{G} = (\dot{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon} + \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}) - \rho_0 \dot{\psi}$$

Isothermal case

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho_0 \dot{\psi} \geq 0$$



$$\boxed{\mathcal{D} = \rho_0 \dot{G} - \boldsymbol{\varepsilon} : \dot{\boldsymbol{\sigma}} \geq 0}$$

Legendre transform

- Given $\psi(x)$ define

$$\left\{ \begin{array}{l} G(x, y) = xy - \psi(x) \\ y \equiv \frac{d\psi(x)}{dx} \end{array} \right. \rightarrow \frac{\partial G(x, y)}{\partial x} = y - \underbrace{\frac{d\psi(x)}{dx}}_{=y} = 0 \rightarrow G(x, \cancel{y}) = G(y)$$

$$y \equiv \frac{d\psi(x)}{dx} \rightarrow x = x(y) \text{ (implicit function)}$$

$$G(y) = x(y)y - \psi(x(y)) \rightarrow \frac{dG(y)}{dy} = \cancel{\frac{dx(y)}{dy}} y + x(y) - \cancel{\frac{d\psi(x)}{dx}} \cancel{\frac{dx(y)}{dy}} = x$$

- Legendre transform

$$\boxed{\left\{ \begin{array}{l} \psi(x) \leftrightarrow G(y) ; \quad G(y) = xy - \psi(x) \\ x \leftrightarrow y ; \quad y \equiv \frac{d\psi(x)}{dx} ; \quad x \equiv \frac{dG(y)}{dy} \end{array} \right.}$$

Elastic material model (stress driven)

- Variable sets definition

- Free variables: $\mathcal{F} := \{\sigma\}$

- Internal variables: $\mathcal{I} := \{\emptyset\}$

- Dependent: $\mathcal{D} := \{\varepsilon(\sigma), G(\sigma)\}$

$$\begin{cases} \dot{\varepsilon} = \frac{\partial \varepsilon(\sigma)}{\partial \sigma} : \dot{\sigma} \\ \dot{G} = \frac{\partial G(\sigma)}{\partial \sigma} : \dot{\sigma} \end{cases}$$

- Potential

- Gibbs energy :

$$\rho_0 G(\sigma) = \sigma : \varepsilon(\sigma) - \rho_0 \psi(\varepsilon(\sigma)) \quad ; \quad \sigma = \frac{\partial \rho_0 \psi(\varepsilon)}{\partial \varepsilon}$$

$$\rightarrow \frac{\partial \rho_0 G(\sigma)}{\partial \sigma} = \varepsilon(\sigma)$$

- Dissipation Isothermal case

$$\mathcal{D} = \rho_0 \dot{G} - \varepsilon : \dot{\sigma} \geq 0$$

$$\mathcal{D} = \underbrace{\left(\frac{\partial \rho_0 G(\sigma)}{\partial \sigma} - \varepsilon \right)}_{= \varepsilon} : \dot{\sigma} = 0 \quad \forall \dot{\sigma}$$

Constitutive equation

$$\varepsilon = \overbrace{\frac{\partial \rho_0 G(\sigma)}{\partial \sigma}}$$

Dissipation

$$\overbrace{\mathcal{D} = 0}$$

Elastic material model (stress driven)

- Linear elastic case:

- Gibbs energy :

$$\begin{cases} \rho_0 \psi(\boldsymbol{\varepsilon}) = \frac{1}{2} \boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon} \rightarrow \boldsymbol{\sigma} = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : \boldsymbol{\varepsilon} \rightarrow \boldsymbol{\varepsilon}(\boldsymbol{\sigma}) = \mathbb{C}^{-1} : \boldsymbol{\sigma} \\ \rho_0 \psi(\boldsymbol{\sigma}) = \frac{1}{2} \underbrace{\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}}_{\boldsymbol{\sigma} : \mathbb{C}^{-1}} = \frac{1}{2} \boldsymbol{\sigma} : \underbrace{\mathbb{C}^{-1} : \mathbb{C} : \mathbb{C}^{-1}}_I : \boldsymbol{\sigma} = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} \\ \rho_0 G(\boldsymbol{\sigma}) = \boldsymbol{\sigma} : \underbrace{\boldsymbol{\varepsilon}(\boldsymbol{\sigma})}_{\mathbb{C}^{-1} : \boldsymbol{\sigma}} - \underbrace{\frac{1}{2} (\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma})}_{\rho_0 \psi(\boldsymbol{\varepsilon}(\boldsymbol{\sigma}))} = \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} - \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} \end{cases}$$

Constitutive equation

$$\boldsymbol{\varepsilon} = \frac{\partial G \rho_0(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}} = \mathbb{C}^{-1} : \boldsymbol{\sigma}$$

Dissipation

$$\mathcal{D} = 0$$

Gibbs potential

$$G(\boldsymbol{\sigma}) = \frac{1}{2} \boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}$$

Inelastic Material

□ Variable set definition

■ Free variables: $\mathbb{F} := \{\sigma\}$

■ Internal variables: $\mathbb{I} := \{q\} \rightarrow \dot{q} = \zeta(\sigma, q)$

■ Dependent: $\mathbb{D} := \{\varepsilon(\sigma, q), G(\sigma, q)\} \rightarrow \dot{G} = \frac{\partial G(\sigma, q)}{\partial \sigma} : \dot{\sigma} + \frac{\partial G(\sigma, q)}{\partial q} \dot{q}$

□ Potential

■ Gibbs energy: $\rho_0 G = \frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma + \eta(q)$

□ Dissipation

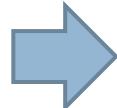
$$\mathcal{D} = \rho_0 \dot{G} - \varepsilon : \dot{\sigma} \geq 0 \rightarrow \mathcal{D} = \underbrace{\left(\frac{\partial \rho_0 G(\sigma, q)}{\partial \sigma} - \varepsilon \right) : \dot{\sigma}}_{f(\sigma, q)} + \underbrace{\frac{\partial G(\sigma, q)}{\partial q} \dot{q}}_{g(\sigma, q)} \geq 0 \quad \forall \dot{\sigma}$$

Inelastic Material

- Using the Coleman's method:

$$f(\sigma, q) = \frac{\partial \rho_0 G(\sigma, q)}{\partial \sigma} - \varepsilon = 0$$

$$g(\sigma, q) = \frac{\partial \rho_0 G(\sigma, q)}{\partial q} \dot{q} \geq 0$$



$$\varepsilon = \frac{\partial \rho_0 G(\sigma, q)}{\partial \sigma}$$

$$\mathcal{D} = \frac{\partial \rho_0 G(\sigma, q)}{\partial q} \dot{q} \geq 0$$

Differentiating the Gibbs potential $\rho_0 G = \frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma + \eta(q)$

Constitutive equation

$$\varepsilon = \underbrace{\frac{\partial \rho_0 G(\sigma, q)}{\partial \sigma}}_{\mathcal{C}^{-1} : \sigma} = \mathbb{C}^{-1} : \sigma$$

Dissipation

$$\mathcal{D} = \overbrace{\frac{\partial \rho_0 G(\sigma, q)}{\partial q} \dot{q}}^{\text{Dissipation}} = \underbrace{\eta'(q)}_{\alpha(q)} \dot{q} \geq 0$$

$$\alpha(q) = \eta'(q) = \frac{d\eta(q)}{dq} \rightarrow \begin{cases} \text{conjugate} \\ \text{internal variable} \end{cases}$$

Inelastic models based on additive stress decomposition (plastic material)

- The total stress is decomposed into elastic and inelastic counterparts

$$\sigma = \underbrace{\sigma^e}_{\text{Elastic stress}} - \underbrace{\sigma^i}_{\text{Inelastic stress}}$$

- Isothermal case:

$\Delta\theta = 0$ No thermal influence

- Variable set definition

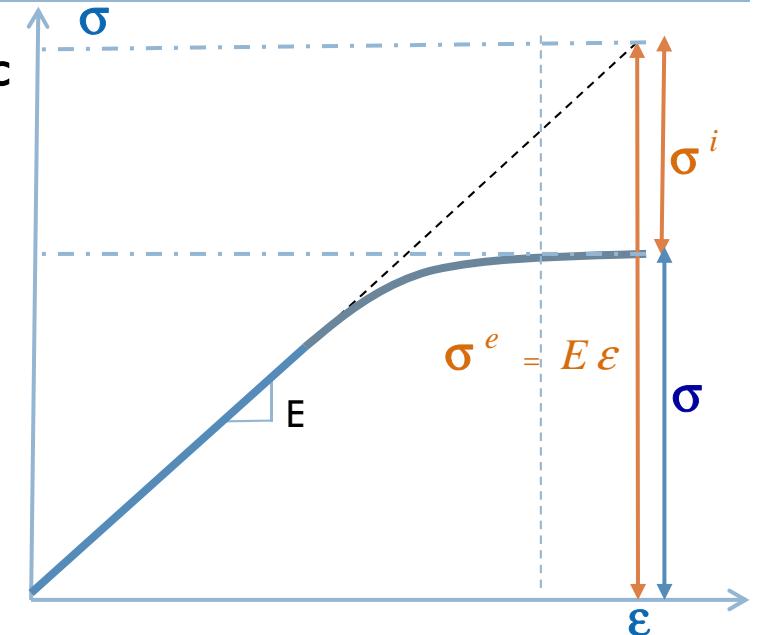
Free variables: $\mathbb{F} := \{\sigma\}$

Internal variables: $\mathbb{I} := \{\sigma^i, q\}$ and evolution equations

Dependent variables:

$$\dot{\sigma}^i = \dot{\sigma}(\sigma, \sigma^i, q) \quad \dot{q} = \dot{q}(\sigma, \sigma^i, q)$$

$$\mathbb{D} := \{\boldsymbol{\varepsilon}(\sigma, \sigma^i, q), G(\sigma, \sigma^i, q)\} \rightarrow \begin{cases} \dot{\boldsymbol{\varepsilon}} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \sigma} : \dot{\sigma} + \frac{\partial \boldsymbol{\varepsilon}}{\partial \sigma^i} : \dot{\sigma}^i + \frac{\partial \boldsymbol{\varepsilon}}{\partial q} \dot{q} \\ \dot{G} = \frac{\partial G}{\partial \sigma} : \dot{\sigma} + \frac{\partial G}{\partial \sigma^i} : \dot{\sigma}^i + \frac{\partial G}{\partial q} \dot{q} \end{cases}$$



Inelastic models based on additive stress decomposition (plastic material)

- Potential

- Gibbs energy: $\rho_0 G(\sigma, \sigma^i, q) = \frac{1}{2} \sigma^e : \mathbb{C}^{-1} : \underbrace{\sigma^e}_{(\sigma + \sigma^i)} + \eta(q)$

- Dissipation: $\mathcal{D} = \rho_0 \dot{G} - \varepsilon : \dot{\sigma} \geq 0$

Replacing the free energy, into the dissipation yields:

$$\mathcal{D} = \underbrace{\left(\frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma} - \varepsilon \right) : \dot{\sigma}}_{f(\sigma, \sigma^i, q)} + \underbrace{\frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma^{(e)}} : \dot{\sigma}^i}_{g_1(\sigma, \sigma^i, q)} + \underbrace{\frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial q} \dot{q}}_{g_2(\sigma, \sigma^i, q)} \geq 0 \quad \forall \dot{\sigma}$$

$g(\sigma, \sigma^i, q)$

REMARK

The following expressions are used:

$$\frac{\partial \rho_0 G}{\partial \sigma} = \frac{\partial \rho_0 G}{\partial \sigma^e} : \underbrace{\frac{\partial \sigma^e}{\partial \sigma}}_{=1} \quad \text{and} \quad \frac{\partial \rho_0 G}{\partial \sigma^i} = \frac{\partial \rho_0 G}{\partial \sigma^e} : \underbrace{\frac{\partial \sigma^e}{\partial \sigma^i}}_{=1}$$

$$\mathcal{D} = f(\sigma, \sigma^i, q) : \dot{\sigma} + g(\sigma, \sigma^i, q) \geq 0 \quad \forall \dot{\sigma}$$

Plastic Material

- Using the Coleman's method for the thermo-elastic material model:

$$f(\sigma, \sigma^i, q) = \frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma^e} - \varepsilon = 0 \quad \rightarrow \quad \varepsilon = \frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma^e} = \frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma}$$

$$g(\sigma, \sigma^i, q) = \underbrace{\frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma^e}}_{=\varepsilon} : \dot{\sigma}^i + \frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial q} \dot{q} \geq 0$$

differentiating the Gibbs potential $\rho_0 G(\sigma, \sigma^i, q) = \frac{1}{2} \sigma^e : \mathbb{C}^{-1} : \sigma^e + \eta(q)$

Constitutive equation

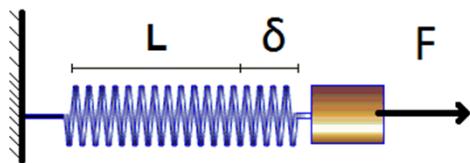
$$\varepsilon = \frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial \sigma^e} = \mathbb{C}^{-1} : \overbrace{(\sigma + \sigma^i)}^{\sigma^e}$$

Dissipation \mathcal{D}

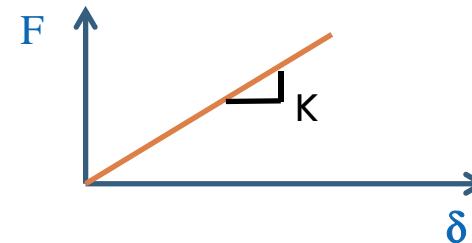
$$\mathcal{D} = \varepsilon : \dot{\sigma}^i + \underbrace{\frac{\partial \rho_0 G(\sigma, \sigma^i, q)}{\partial q} \dot{q}}_{\eta'(q) := \alpha(q)} \geq 0 : \dot{\sigma} = \overbrace{\varepsilon : \dot{\sigma}^i + \underbrace{\alpha(q)}_{\text{Conjugate internal variable}} \dot{q}} \geq 0$$

Rheological models

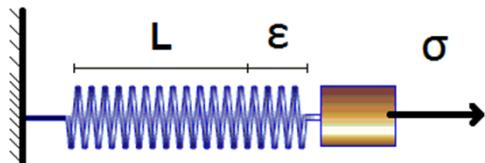
- Elastic element



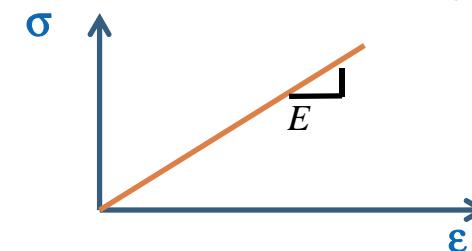
$$F = K \delta$$



- Elastic model



$$\sigma = E \epsilon$$



- Stress driven model $\mathcal{F} := \{\sigma\}, \mathcal{I} := \{\emptyset\}$ and $\mathcal{D} := \{\epsilon(\sigma, \beta)\}, G(\sigma, \beta)\}$

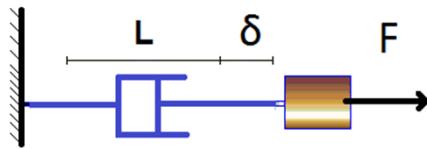
$$\rho_0 G(\sigma) = \frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma \Rightarrow \boxed{\mathcal{D} = 0}$$

- Strain driven model $\mathcal{F} := \{\epsilon\}, \mathcal{I} := \{\emptyset\}$ and $\mathcal{D} := \{\sigma, \psi\}$

$$\rho_0 \psi(\epsilon) = \frac{1}{2} \epsilon : \mathbb{C} : \epsilon \Rightarrow \boxed{\mathcal{D} = 0}$$

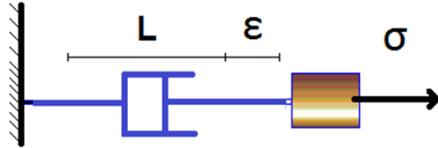
Rheological models

- Viscous element



$$F = \eta \dot{\delta} \rightarrow \dot{\delta} = \frac{1}{\eta} F$$

- Viscous model



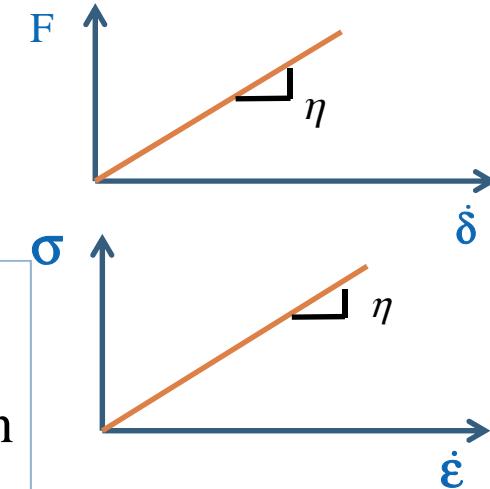
$$\begin{cases} \beta = \epsilon \rightarrow \text{Internal variable} \\ \dot{\beta} = \frac{1}{\eta} \sigma \rightarrow \text{Evolution equation} \end{cases}$$

- Stress driven model: $\mathcal{F} := \{\sigma\}$ $\mathcal{I} := \{\beta\}$ $\mathcal{D} := \{\epsilon, G\}$

$$\begin{cases} \rho_0 G(\sigma, \beta) = \sigma : \beta \\ \epsilon = \frac{\partial \rho_0 G(\sigma, \beta)}{\partial \sigma} = \beta \end{cases} \xrightarrow{\quad \mathcal{D} = \underbrace{\frac{\partial \rho_0 G(\sigma, \beta)}{\partial \beta}}_{= \sigma} : \dot{\beta} = \sigma : \dot{\beta} = \frac{1}{\eta} \underbrace{\sigma}_{\geq 0} \geq 0 \quad} \eta \geq 0$$

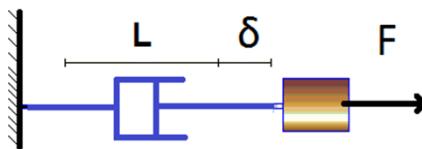
REMARK

Positive dissipation (2nd. principle) translates into restrictions on the physically meaningful values of the material properties.



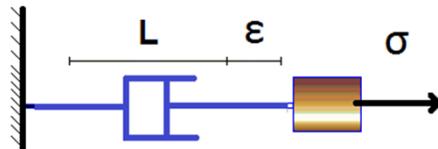
Rheological Models

- Viscous element

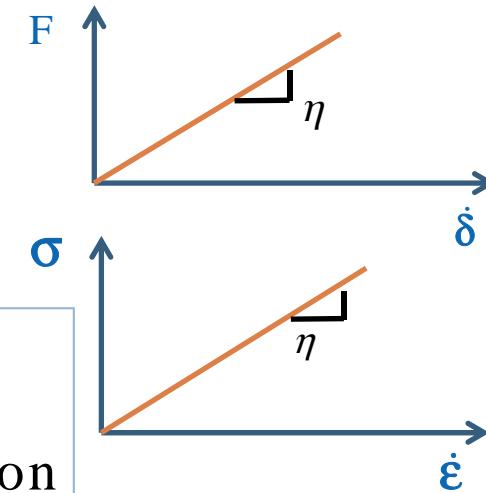


$$F = \eta \dot{\delta} \rightarrow \dot{\delta} = \frac{1}{\eta} F$$

- Viscous model



$$\begin{cases} \beta = \epsilon \rightarrow \text{Internal variable} \\ \dot{\beta} = \frac{1}{\eta} \sigma \rightarrow \text{Evolution equation} \end{cases}$$



Free and internal variables
must be different !!!

- Strain driven model $\mathcal{F} := \{\epsilon\}$

$$\overbrace{\mathcal{I}}^{\substack{\beta \\ = \epsilon}} := \overbrace{\{\beta\}}^{\substack{\epsilon \\ = \epsilon}} = \mathcal{F}$$

$$\mathcal{D} := \{\sigma, \psi\}$$

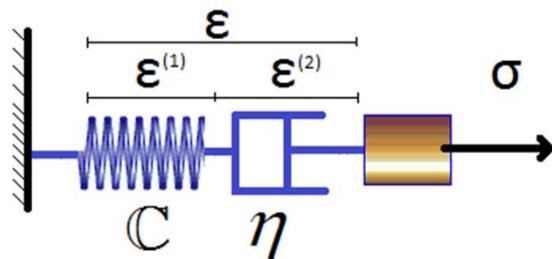
$$\left\{ \begin{array}{l} \rho_0 \psi = \rho_0 G - \sigma : \epsilon = \sigma : \underbrace{\beta}_{\epsilon} - \sigma : \epsilon = 0 \rightarrow \text{Free energy does not exist} \\ \sigma = \frac{\partial \rho_0 \psi(\epsilon, \beta)}{\partial \epsilon} \end{array} \right.$$



A strain driven version of the model has
no sense

Maxwell's model

- Viscous Model



Let's define

$$\left\{ \begin{array}{l} \boldsymbol{\varepsilon} = \underbrace{\boldsymbol{\varepsilon}^e}_{\boldsymbol{\varepsilon}^{(1)}} + \underbrace{\boldsymbol{\varepsilon}^i}_{\boldsymbol{\varepsilon}^{(2)}} \\ \boldsymbol{\varepsilon}^i = \boldsymbol{\beta} \end{array} \right.$$

being

$$\left\{ \begin{array}{l} \boldsymbol{\varepsilon}^e = \mathbb{C}^{-1} : \boldsymbol{\sigma} \\ \dot{\boldsymbol{\varepsilon}}^i = \frac{1}{\eta} \boldsymbol{\sigma} \end{array} \right.$$

- Strain driven model

- Variable sets definition

$$\mathbb{F} := \{\boldsymbol{\varepsilon}\}, \quad \mathbb{I} := \{\boldsymbol{\beta}\} \text{ and } \mathbb{D} := \{\boldsymbol{\sigma}, \psi\} \quad \dot{\boldsymbol{\beta}} = \frac{1}{\eta} \boldsymbol{\sigma}$$

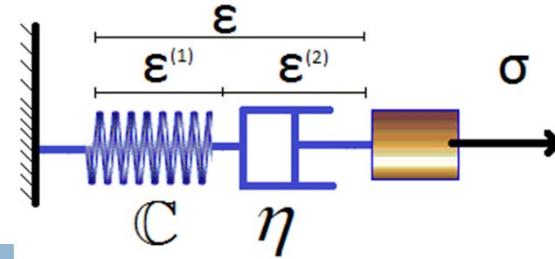
- Potential $\rho_0 \psi(\boldsymbol{\varepsilon}) = \rho_0 \psi^{(1)}(\boldsymbol{\varepsilon}) + \rho_0 \psi^{(2)}(\boldsymbol{\varepsilon})$

Elastic $\rho_0 \psi^{(1)}(\boldsymbol{\varepsilon}, \boldsymbol{\beta}) = \frac{1}{2} \boldsymbol{\varepsilon}^{(1)} : \mathbb{C} : \boldsymbol{\varepsilon}^{(1)} = \frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\beta}) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta})$

Viscous $\rho_0 \psi^{(2)}(\boldsymbol{\varepsilon}) = 0$

$$\rho_0 \psi(\boldsymbol{\varepsilon}) = \rho_0 \psi^{(1)}(\boldsymbol{\varepsilon}, \boldsymbol{\beta}) + \rho_0 \psi^{(2)}(\boldsymbol{\varepsilon}) = \frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\beta}) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta})$$

Maxwell's model



- Constitutive equation $\rho_0 \psi(\boldsymbol{\varepsilon}) = \frac{1}{2} (\boldsymbol{\varepsilon} - \boldsymbol{\beta}) : \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta})$

$$\sigma = \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\beta})}{\partial \boldsymbol{\varepsilon}} = \mathbb{C} : (\underbrace{\boldsymbol{\varepsilon} - \boldsymbol{\beta}}_{\boldsymbol{\varepsilon}^e}) \quad \Rightarrow \quad \dot{\sigma} = \mathbb{C} : (\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\beta}}) = \mathbb{C} : (\dot{\boldsymbol{\varepsilon}} - \frac{1}{\eta} \sigma) \quad \Rightarrow$$

- Dissipation

Constitutive equation

$$\dot{\boldsymbol{\varepsilon}} = \overbrace{\mathbb{C}^{-1} : \dot{\sigma}} + \frac{1}{\eta} \sigma$$

$$\mathcal{D} = - \frac{\partial \rho_0 \psi(\boldsymbol{\varepsilon}, \boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \dot{\boldsymbol{\beta}} = -[-\frac{1}{2} \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta})] : \dot{\boldsymbol{\beta}} = \frac{1}{2} \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta}) : \dot{\boldsymbol{\beta}} \geq 0$$

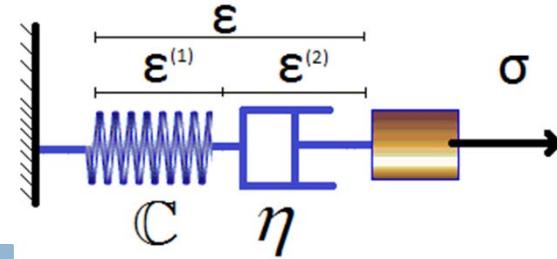
$$\left. \begin{aligned} \boldsymbol{\sigma} &= \mathbb{C} : \boldsymbol{\varepsilon}^e = \mathbb{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\beta}) \\ \boldsymbol{\sigma} &= \eta \dot{\boldsymbol{\beta}} \end{aligned} \right\} \rightarrow \mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\beta}} = \eta \underbrace{\dot{\boldsymbol{\beta}} : \dot{\boldsymbol{\beta}}}_{\geq 0} \geq 0 \quad \Rightarrow \boxed{\eta \geq 0}$$

- Stress driven model

- Variable sets definition

$$\mathbb{F} := \{\boldsymbol{\sigma}\} \quad \mathbb{I} := \{\boldsymbol{\beta}\} \quad \mathbb{D} := \{\boldsymbol{\varepsilon}, G\} \quad \dot{\boldsymbol{\beta}} = \frac{1}{\eta} \boldsymbol{\sigma}$$

Maxwell's model



- Potential $\rho_0 G(\sigma, \beta) = \rho_0 G^{(1)}(\sigma, \beta) + \rho_0 G^{(2)}(\sigma, \beta)$

Elastic $\rho_0 G^{(1)}(\sigma, \beta) = \frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma$

Viscous $\rho_0 G^{(2)}(\sigma, \beta) = \sigma : \beta$



$$\rho_0 G(\sigma, \beta) = \frac{1}{2} \sigma : \mathbb{C}^{-1} : \sigma + \sigma : \beta$$

- Constitutive equation

$$\varepsilon = \frac{\partial \rho_0 G(\sigma, \beta)}{\partial \sigma} = \mathbb{C}^{-1} : \sigma + \beta \quad \Rightarrow \quad \dot{\varepsilon} = \mathbb{C}^{-1} : \dot{\sigma} + \underbrace{\frac{\dot{\beta}}{\eta} \sigma}_{= \dot{\beta}}$$

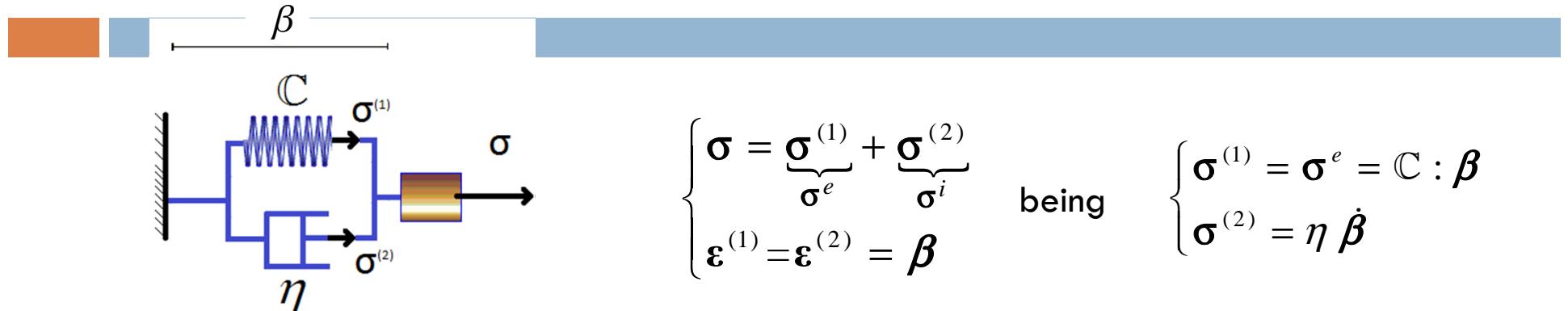
Constitutive equation

$$\dot{\varepsilon} = \mathbb{C}^{-1} : \dot{\sigma} + \frac{1}{\eta} \sigma$$

- Dissipation

$$\mathcal{D} = \frac{\partial \rho_0 G(\sigma, \beta)}{\partial \beta} : \dot{\beta} = \sigma : \underbrace{\frac{\dot{\beta}}{\eta} \sigma}_{= \dot{\beta}} \geq 0 \quad \Rightarrow \quad \mathcal{D} = \frac{1}{\eta} \sigma : \sigma \geq 0 \quad \Rightarrow \quad \eta \geq 0$$

Kelvin's model



$$\left\{ \begin{array}{l} \boldsymbol{\sigma} = \underbrace{\boldsymbol{\sigma}^{(1)}}_{\boldsymbol{\sigma}^e} + \underbrace{\boldsymbol{\sigma}^{(2)}}_{\boldsymbol{\sigma}^i} \\ \boldsymbol{\varepsilon}^{(1)} = \boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\beta} \end{array} \right. \quad \text{being} \quad \left\{ \begin{array}{l} \boldsymbol{\sigma}^{(1)} = \boldsymbol{\sigma}^e = \mathbb{C} : \boldsymbol{\beta} \\ \boldsymbol{\sigma}^{(2)} = \eta \dot{\boldsymbol{\beta}} \end{array} \right.$$

- Stress driven model

- Variable set definition

$$\mathbb{F} := \{\boldsymbol{\sigma}\} \quad \text{and} \quad \mathbb{I} := \{\boldsymbol{\beta}\} \quad \dot{\boldsymbol{\beta}} = \frac{1}{\eta} \boldsymbol{\sigma}^{(2)} = \frac{1}{\eta} (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(1)}) = \frac{1}{\eta} (\boldsymbol{\sigma} - \mathbb{C} : \boldsymbol{\beta})$$

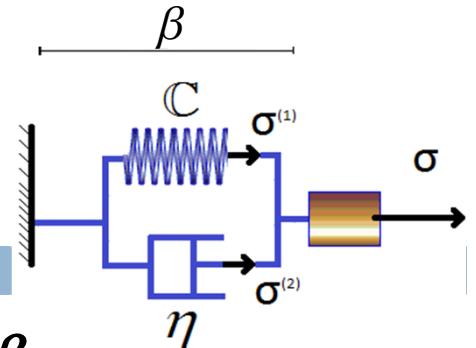
- Potential $\rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \rho_0 G^{(1)}(\boldsymbol{\sigma}, \boldsymbol{\beta}) + \rho_0 G^{(2)}(\boldsymbol{\sigma}, \boldsymbol{\beta})$

Elastic $\rho_0 G^{(1)}(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \frac{1}{2} \boldsymbol{\sigma}^{(1)} : \mathbb{C}^{-1} : \boldsymbol{\sigma}^{(1)} = \frac{1}{2} (\underbrace{\boldsymbol{\beta} : \mathbb{C}}_{\boldsymbol{\sigma}^{(1)}}) : \mathbb{C}^{-1} : (\underbrace{\mathbb{C} : \boldsymbol{\beta}}_{\boldsymbol{\sigma}^{(1)}}) = \frac{1}{2} \boldsymbol{\beta} : \mathbb{C} : \boldsymbol{\beta}$

Viscous $\rho_0 G^{(2)}(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \boldsymbol{\sigma}^{(2)} : \boldsymbol{\beta} = (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{(1)}) : \boldsymbol{\beta} = \boldsymbol{\sigma} : \boldsymbol{\beta} - \underbrace{\boldsymbol{\beta} : \mathbb{C}}_{\boldsymbol{\sigma}^{(1)}} : \boldsymbol{\beta}$

$$\rho_0 G(\boldsymbol{\sigma}, \boldsymbol{\beta}) = \rho_0 G^{(1)} + \rho_0 G^{(2)} = \boldsymbol{\sigma} : \boldsymbol{\beta} - \frac{1}{2} \boldsymbol{\beta} : \mathbb{C} : \boldsymbol{\beta}$$

Kelvin's model



- Constitutive equation $\rho_0 G(\sigma, \beta) = \sigma : \beta - \frac{1}{2} \beta : \mathbb{C} : \beta$

$$\varepsilon = \frac{\partial \rho_0 G(\sigma, \beta)}{\partial \sigma} = \beta$$

- Dissipation

$$\mathcal{D} = \frac{\partial \rho_0 G(\sigma, \beta)}{\partial \beta} : \dot{\beta} = (\underbrace{\sigma - \mathbb{C} : \beta}_{=\sigma^{(1)}} : \dot{\beta}) : \dot{\beta} = \eta \underbrace{\dot{\beta} : \dot{\beta}}_{>0} \geq 0 \rightarrow \boxed{\eta \geq 0}$$

- Strain driven model

- Variables set definition

$$\mathbb{F} := \{\boldsymbol{\varepsilon}\}, \quad \mathbb{I} := \{\boldsymbol{\beta}\}$$

$$\mathbb{D} := \{\boldsymbol{\sigma}, \psi\}$$

$$\dot{\beta} = \frac{1}{\eta} \boldsymbol{\sigma}^{(2)}$$

NOT PHYSICAL

Free and internal variables
must be different !!!!



END OF LECTURE 2