

CH.2. CONTINUUM DAMAGE MODELS

Computational Solid Mechanics- Xavier Oliver-UPC

2.1 Introduction

Ch.1. Continuum Damage Models

Continuum Damage Models

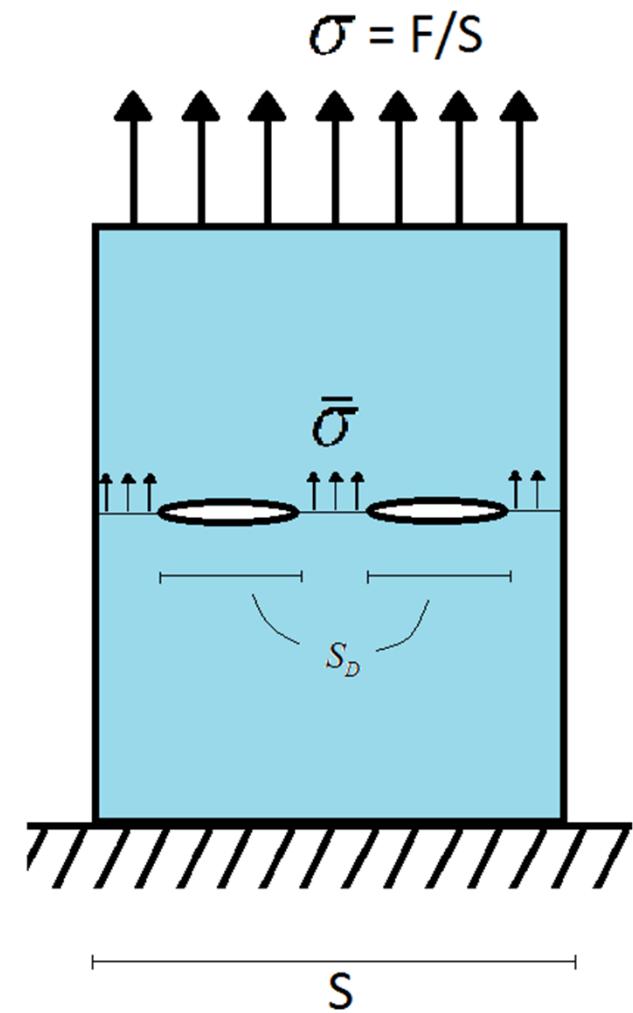
Phenomenological approach

- Damaged section: S^d
- Effective section: $\bar{S} = S - S^d$
- Effective stress: $\bar{\sigma} = E\varepsilon$
 - Equilibrium $F = \bar{\sigma} \bar{S} = \sigma S$
- Apparent (observed) stress σ

$$\sigma = \frac{\bar{S}}{S} \bar{\sigma} = \frac{S - S_d}{S} \bar{\sigma} = \left(1 - \frac{S_d}{S}\right) \bar{\sigma} = \underbrace{\left(1 - \frac{S_d}{S}\right)}_d E \varepsilon$$

$$\sigma = (1 - d) E \varepsilon$$

Damage parameter



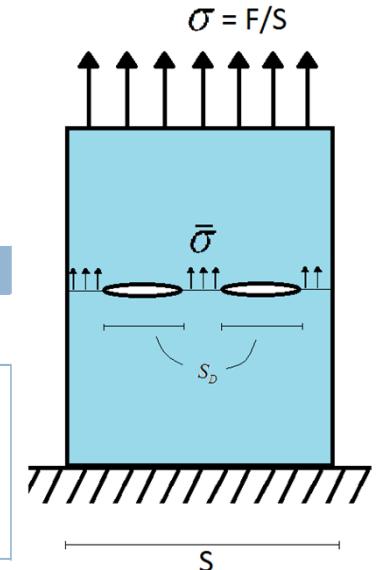
Damage parameter

□ Physical Interpretation

$$d(t) = d(S_d(t)) = \frac{S_d(t)}{S}$$

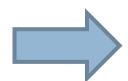
Remark

Ratio of the damaged area over the total area at a local material point



□ Limits of the damage parameter

$$0 \leq S_d \leq S$$



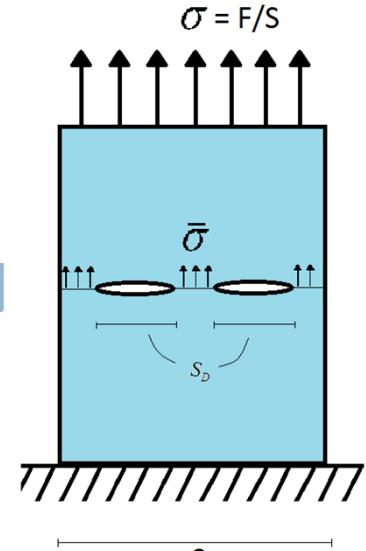
$$0 \leq d \leq 1$$

$$\left. \begin{array}{l} d(0) = \frac{0}{S} = 0 \\ d(S_d) = \frac{S_d}{S} = 1 \end{array} \right\} \begin{array}{l} \text{Undamaged state} \\ \text{Fully damaged state} \end{array}$$

Remark

d is a scalar variable that accounts for the micro-cracks or micro-voids independently of their orientation (isotropic damage models)

Damage parameter

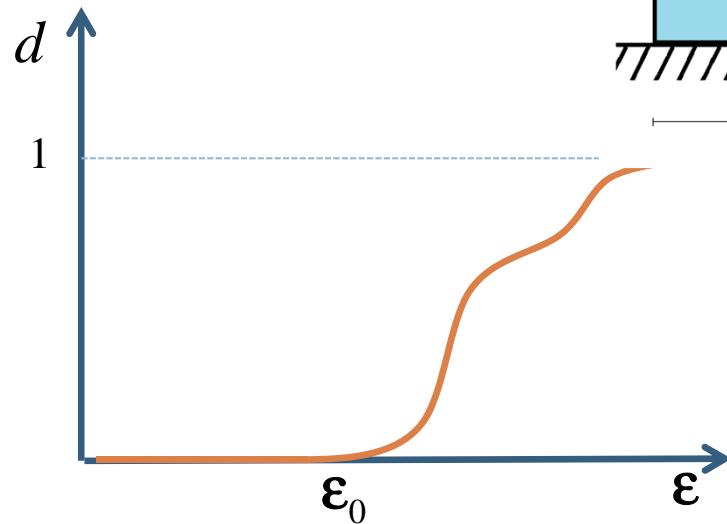


- Properties

- Damage must always increase

$$d(t) = \frac{S_d(t)}{S} \Rightarrow \dot{d} = \frac{\dot{S}_d(t)}{S} \geq 0$$

2nd Law of Thermodynamics

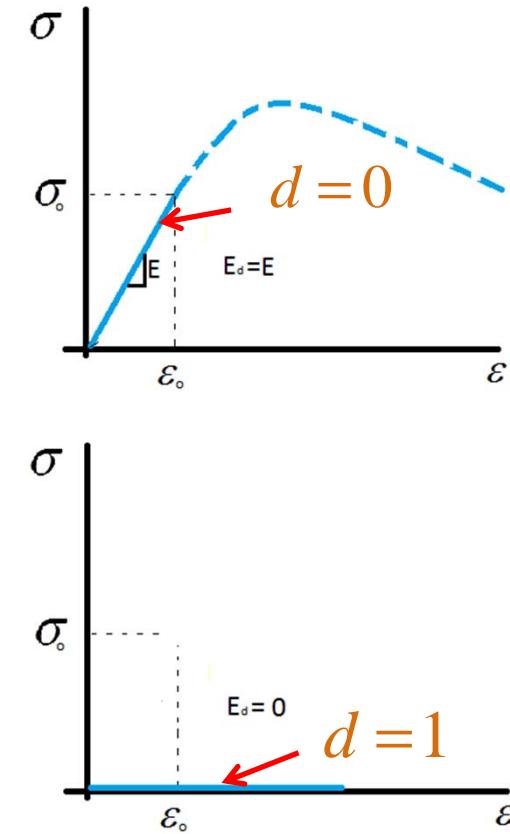
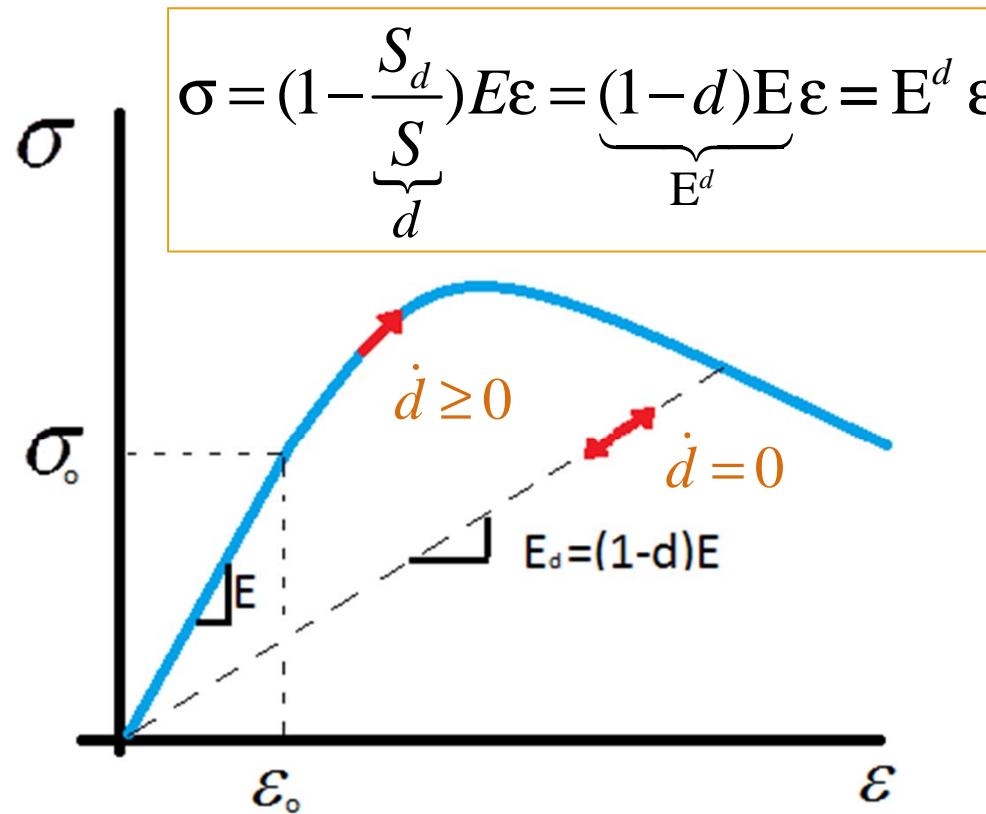


- Damage or (degradation) is initiated when the strain (or stress) exceeds the initial damage threshold (ϵ_0, σ_0)

$$d = 0 \quad \text{if} \quad \epsilon \leq \epsilon_0 \quad \text{or} \quad \sigma \leq \sigma_0$$

Continuum Damage Models

□ 1D Stress-Strain damage constitutive model



Remark

Unloading does not produce healing ($\dot{d} = 0$)

Continuum Damage Models

$$\sigma = \underbrace{(1-d)E}_{E_d} \varepsilon = E_d \varepsilon$$

- 3D Stress-Strain

- The constitutive equation takes the following form:

$$\begin{cases} \sigma = \underbrace{(1-d)\mathbb{C}}_{\mathbb{C}^d} : \varepsilon = \mathbb{C}^d : \varepsilon = (1-d)\bar{\sigma} \\ \bar{\sigma} = \mathbb{C} : \varepsilon \rightarrow \text{effective stress} \end{cases}$$

$$\mathbb{C}^d = (1-d)\mathbb{C} = (1-d)[\lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbb{I}]$$

- The damage parameter is subjected to the same restrictions:

$$\begin{cases} \dot{d} \geq 0 & \text{Never decreases} \\ 0 \leq d \leq 1 & \text{Admissible range} \end{cases}$$

Constitutive equation in a thermodynamic framework

- Variable space definition:

- The strain is the free variable $\mathcal{F} := \{\boldsymbol{\varepsilon}\}$
 - There is one scalar internal variable: $\mathcal{I} := \{r\}$
 - Evolution equation $\dot{r} = \lambda(\boldsymbol{\varepsilon}, r)$
 - Dependent variables $\mathcal{D} := \{\sigma(\boldsymbol{\varepsilon}, r), \psi(\boldsymbol{\varepsilon}, r), d(\boldsymbol{\varepsilon}, r) \dots\}$

- Potential

- Free energy

$$\psi(\boldsymbol{\varepsilon}, d(r)) = (1 - d(r))\psi_0(\boldsymbol{\varepsilon}) = (1 - d(r)) \frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})$$

$$\dot{\psi} = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial d} \dot{d}$$

Constitutive equation in a thermodynamic framework

$$\dot{\psi} = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}} + \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial d} \dot{d}$$

□ Dissipation $\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$

$$\mathcal{D} = (\boldsymbol{\sigma} - \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}}) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial d} \dot{d} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

$$\psi(\boldsymbol{\varepsilon}, d(r)) = (1 - d(r))\psi_0(\boldsymbol{\varepsilon}) = (1 - d(r)) \underbrace{\frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})}_{\psi_0(\boldsymbol{\varepsilon}) \geq 0}$$

Coleman's method

$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \frac{\partial \psi_0(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \mathbb{C} : \boldsymbol{\varepsilon}$$

$$\mathcal{D} = - \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial d} \dot{d} = \underbrace{\psi_0}_{\geq 0} \dot{d} \geq 0 \quad \rightarrow \quad \boxed{\dot{d} \geq 0}$$

Remark

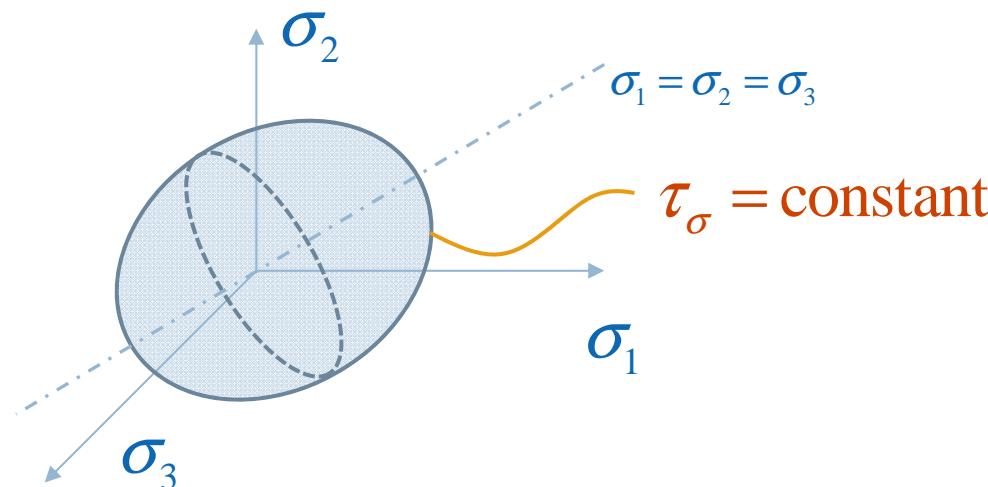
Second principle postulates that healing ($\dot{d} < 0$) is not physically meaningful

Constitutive equation-additional elements: stress/strain norms

□ STRESS NORM

$$\tau_{\sigma} = \sqrt{\boldsymbol{\sigma} : \mathbb{M} : \boldsymbol{\sigma}} = \|\boldsymbol{\sigma}\|_{\mathbb{M}}$$

Defining $\mathbb{M} = \alpha \mathbf{1} \otimes \mathbf{1} + \beta \mathbb{I}$ with $\begin{cases} \alpha + \beta \geq 0 \\ \beta \geq 0 \end{cases}$



Remark

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

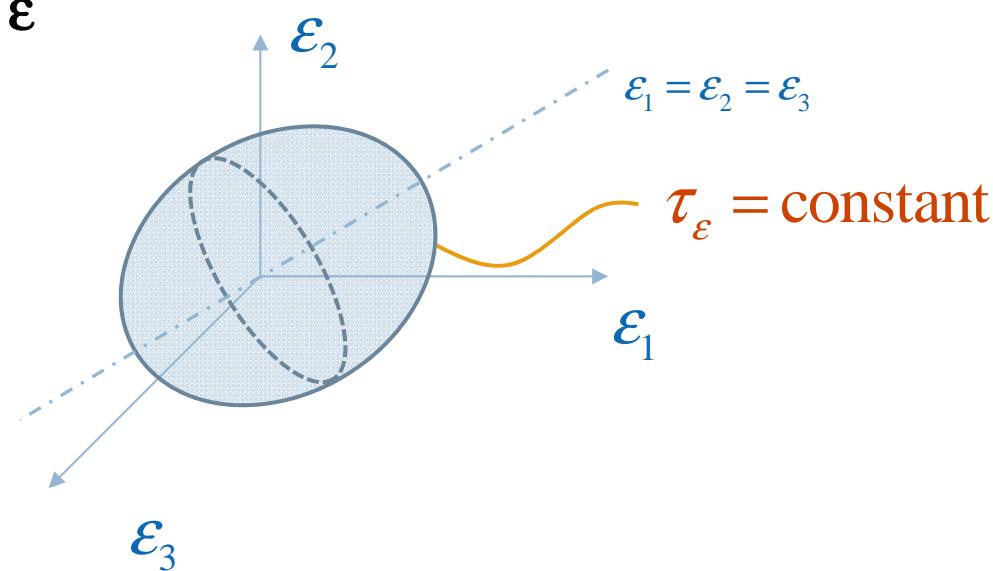
High-Westergaard
“stress space”

Constitutive equation-additional elements: stress/strain norms

□ STRAIN NORM

$$\tau_{\varepsilon} = \|\bar{\sigma}\|_M = \sqrt{\bar{\sigma} : M : \bar{\sigma}} = (\bar{\sigma} : M : \underbrace{\bar{\sigma}}_{C:\varepsilon})^{\frac{1}{2}} = (\varepsilon : \underbrace{C : M : C}_{C^*} : \varepsilon)^{\frac{1}{2}} = \sqrt{\varepsilon : C^* : \varepsilon}$$

$$\tau_{\varepsilon} = \|\varepsilon\|_{C^*} = \sqrt{\varepsilon : C^* : \varepsilon}$$



Constitutive equation-additional elements: stress/strain norms

□ REMARKS

■ 1) Taking

$$\mathbb{M} = \mathbb{C}^{-1}$$



$$\begin{cases} \tau_{\sigma} = \sqrt{\sigma : \mathbb{C}^{-1} : \sigma} = \sqrt{2G_0(\sigma)} \\ \tau_{\varepsilon} = \sqrt{\varepsilon : \mathbb{C} : \varepsilon} = \sqrt{2\psi_0(\varepsilon)} \end{cases}$$

■ 2) Relationship between the norms

$$\begin{cases} \tau_{\sigma} = (\sigma : \mathbb{C}^{-1} : \sigma)^{\frac{1}{2}} = \sqrt{(1-d)^2 \varepsilon : (\mathbb{C} : \mathbb{C}^{-1} : \mathbb{C}) : \varepsilon} = (1-d) \underbrace{\sqrt{\varepsilon : \mathbb{C} : \varepsilon}}_{\tau_{\varepsilon}} = (1-d) \tau_{\varepsilon} \\ \sigma = (1-d) \mathbb{C} : \varepsilon \end{cases}$$

A blue arrow pointing from left to right, indicating a mathematical derivation or consequence.
$$\tau_{\sigma} = (1-d) \tau_{\varepsilon}$$

2.2 Inviscid damage model

Ch.1. Continuum Damage Models

Constitutive equation-additional elements: damage criterion

- **Damage function** (in stress space)

$$f(\sigma, r) = \tau_\sigma - q(r)$$

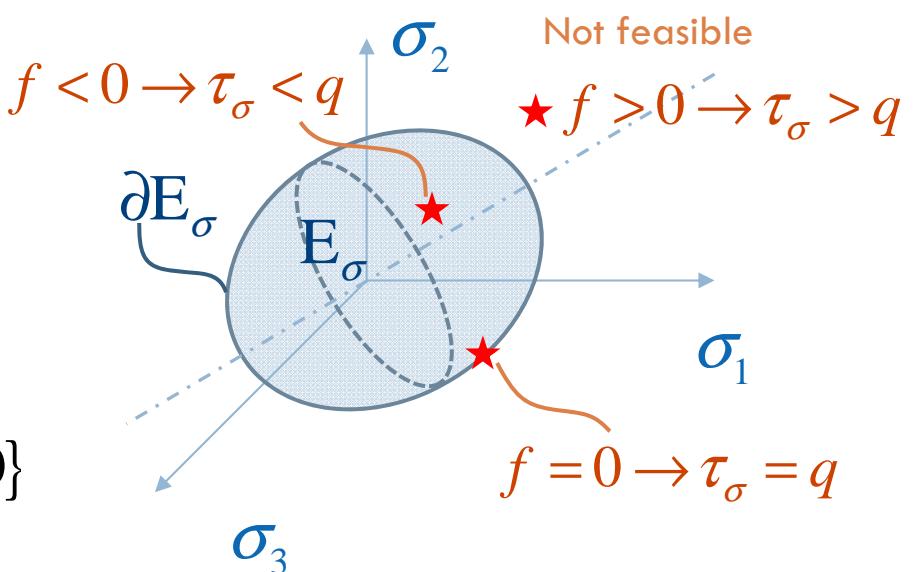
- where $q(r)$ is the **hardening variable**
and controls the size of the elastic
domain

- **Elastic domain** (in stress space)

$$E_\sigma := \{ \sigma \in \mathbb{S} \mid f(\sigma, r) = \tau_\sigma - q(r) < 0 \}$$

- **Damage surface** (in stress space)

$$\partial E_\sigma := \{ \sigma \in \mathbb{S} \mid f(\sigma, r) = \tau_\sigma - q(r) = 0 \}$$



Constitutive equation-additional elements: damage criterion

- **Damage function** (in strain space)

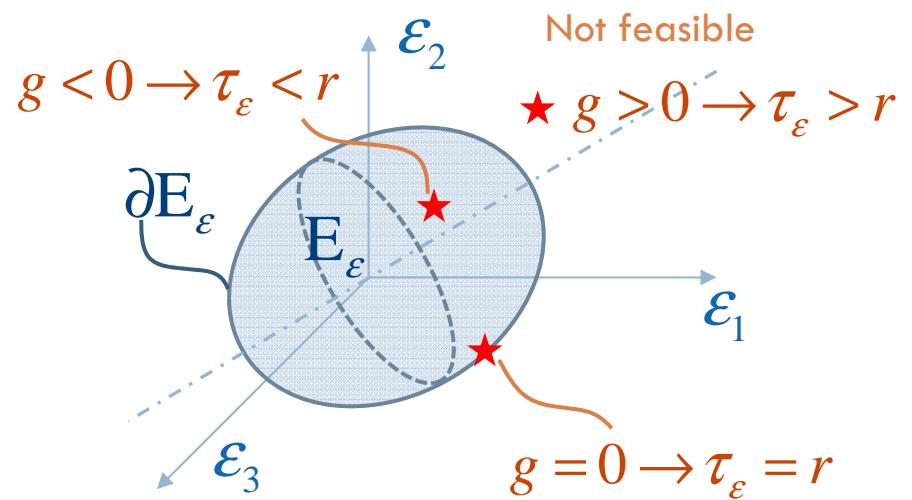
$$g(\varepsilon, r) = \tau_\varepsilon - r$$

- **Elastic domain** (in strain space)

$$E_\varepsilon := \{\varepsilon \in \mathbb{S} \mid g(\varepsilon, r) \equiv \tau_\varepsilon - r < 0\}$$

- **Damage surface** (in strain space)

$$\partial E_\varepsilon := \{\varepsilon \in \mathbb{S} \mid g(\varepsilon, r) \equiv \tau_\varepsilon - r = 0\}$$



Constitutive equation-additional elements: elastic domains relationship

- THEOREM : The elastic domains in stress and strain spaces are equivalent, i.e.:

$$\text{If } \sigma \in E_\sigma \Leftrightarrow \varepsilon \in E_\varepsilon$$

- Proof

$$f(\sigma, r) = \tau_\sigma - q(r) < 0 \quad \Rightarrow \quad E_\sigma := \{ \sigma \in \mathbb{S} \mid f(\sigma, r) = \tau_\sigma - q(r) < 0 \}$$

$$g(\varepsilon, r) = \tau_\varepsilon - r < 0 \quad \Rightarrow \quad E_\varepsilon := \{ \varepsilon \in \mathbb{S} \mid g(\varepsilon, r) = \tau_\varepsilon - r < 0 \}$$

$$\begin{cases} \sigma \in E_\sigma \Leftrightarrow f(\sigma, r) = \tau_\sigma - q(r) = \underbrace{\tau_\sigma}_{(1-d)\tau_\varepsilon} - \underbrace{q(r)}_{(1-d)r} = \underbrace{(1-d)}_{\geq 0} \underbrace{(\tau_\varepsilon - r)}_{g(\varepsilon, r) < 0} < 0 \\ \tau_\sigma = (1-d)\tau_\varepsilon \\ 1-d = \frac{q(r)}{r} \end{cases}$$

$$\sigma \in E_\sigma \Leftrightarrow g(\varepsilon, r) < 0 \Leftrightarrow \varepsilon \in E_\varepsilon$$

Constitutive equation-additional elements: hardening/softening variable

□ Evolution of the internal variables

$$\dot{r} = \lambda(\varepsilon, r) \geq 0, \quad r|_{t=0} = r_0 > 0, \quad r \in [r_0, \infty]$$

Complete damaged state

Material property,
undamaged state

□ Damage variable

$$d := 1 - \frac{q(r)}{r} \quad 0 \leq d \leq 1$$

$$q(r_0) = r_0 \rightarrow d(r_0) = 0$$

□ Hardening parameter

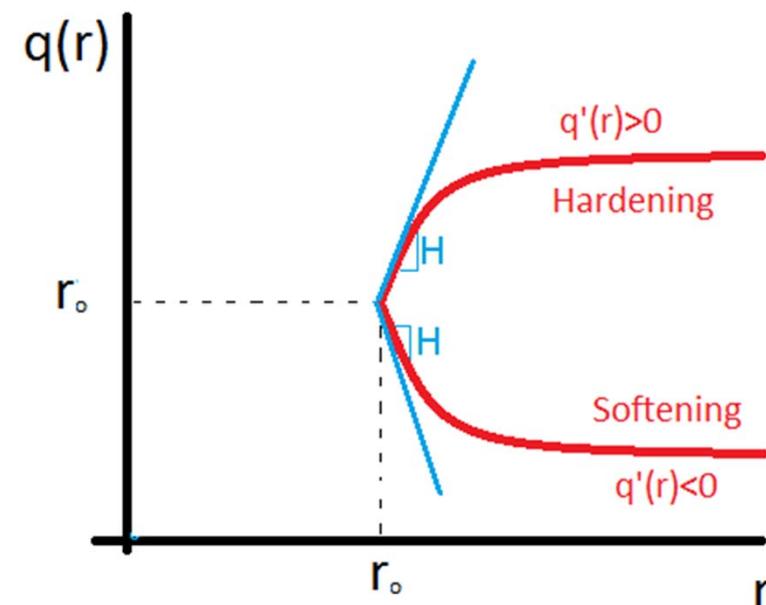
$$H > 0$$

$$H := \frac{dq(r)}{dr} = q'(r)$$

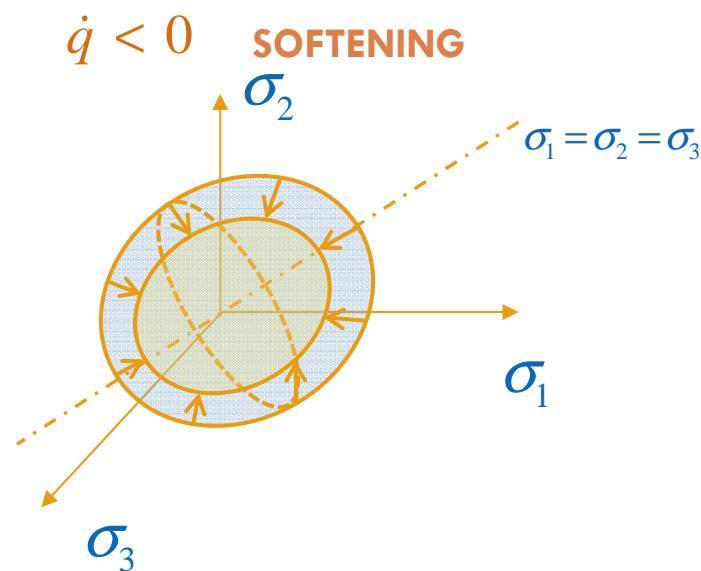
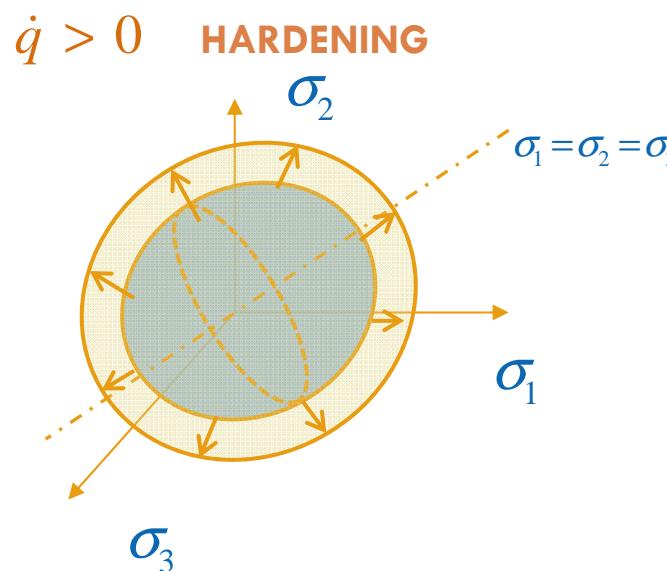
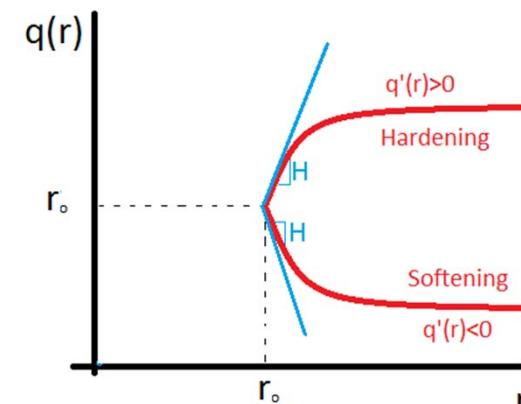
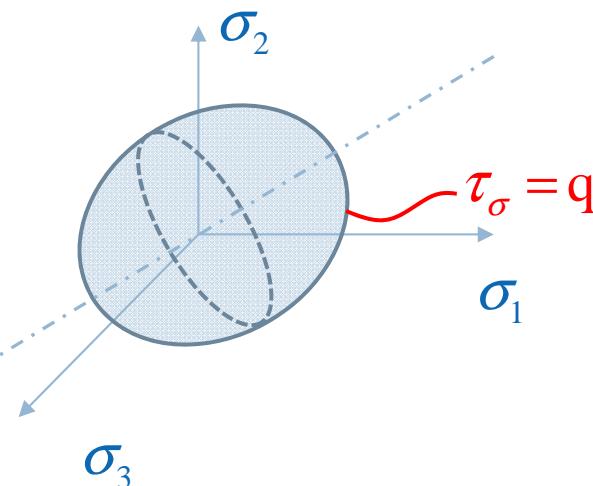
Hardening

$$H < 0$$

Softening



Constitutive equation-additional elements: hardening/softening variable

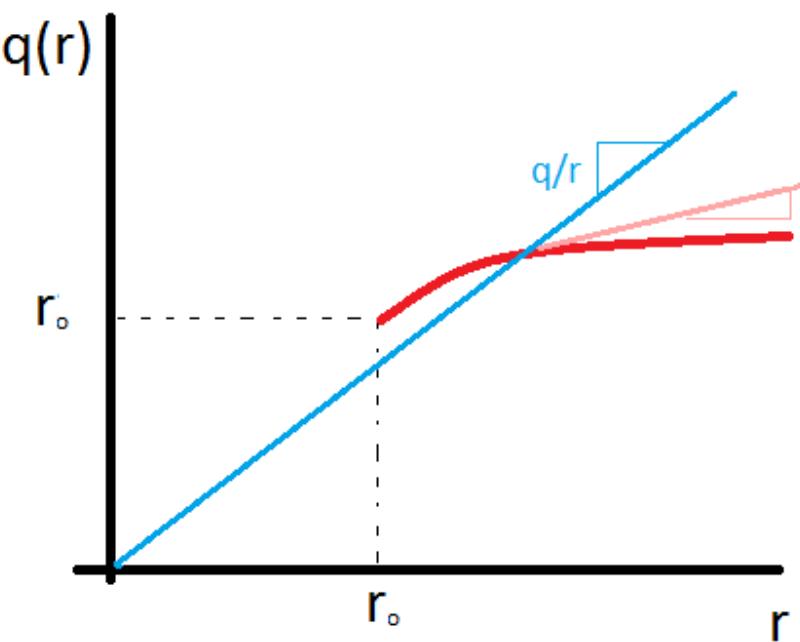


Constitutive equation-additional elements: hardening/softening variable

□ Conditions for the Hardening modulus

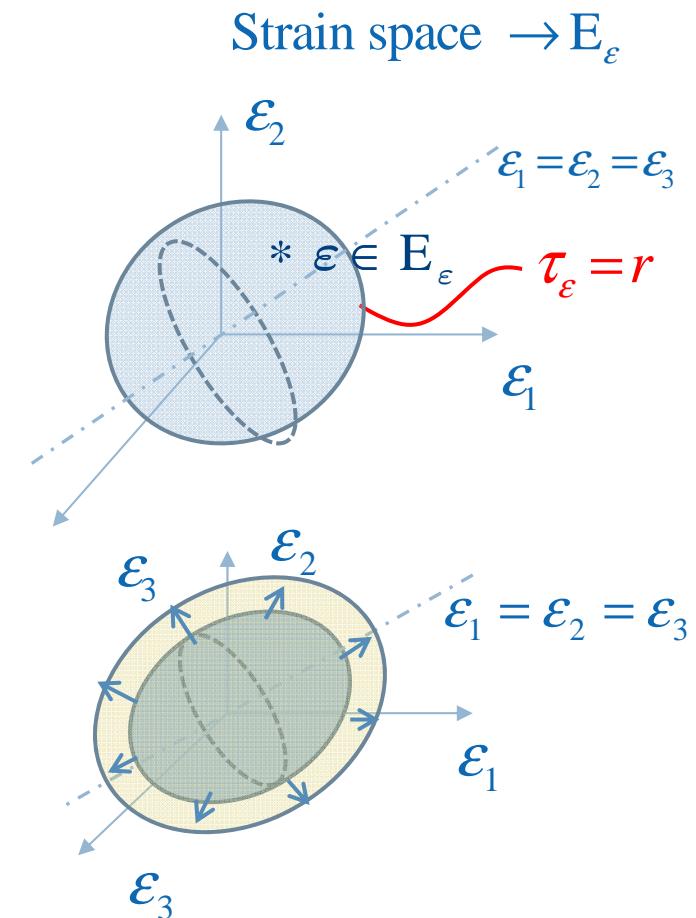
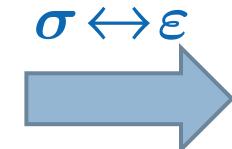
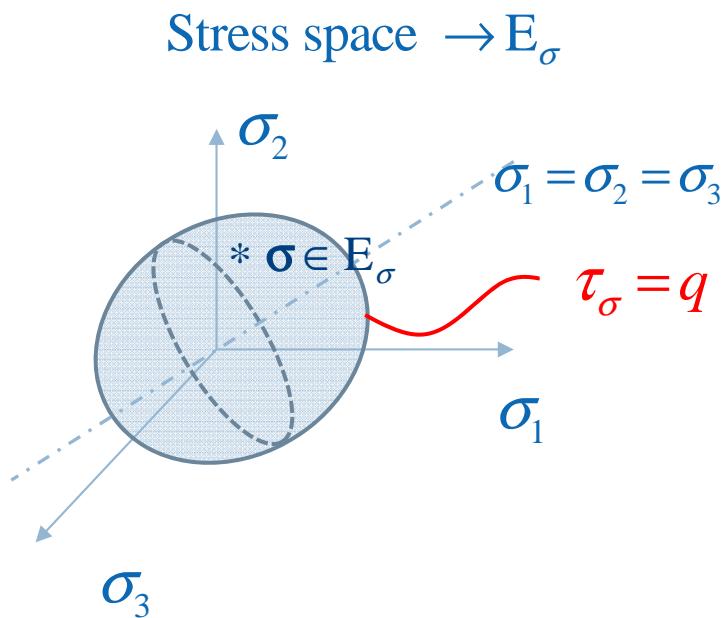
$$\begin{cases} d(r) = 1 - \frac{q(r)}{r} \\ \dot{d} \geq 0 \end{cases} \rightarrow \dot{d} = \frac{q(r) - q'(r)r}{r^2} \geq 0 \rightarrow q(r) \geq \underbrace{q'(r)r}_H$$

$$\rightarrow H \leq \frac{q(r)}{r}$$



Constitutive equation-additional elements: elastic domains relationship

- Elastic domain (in stress and strain spaces)



Remark

$\dot{r} \geq 0 \Rightarrow E_\varepsilon$ always grows, regardless
hardening or softening take place

Constitutive equation-additional elements: Karush/Kuhn/Tucker conditions

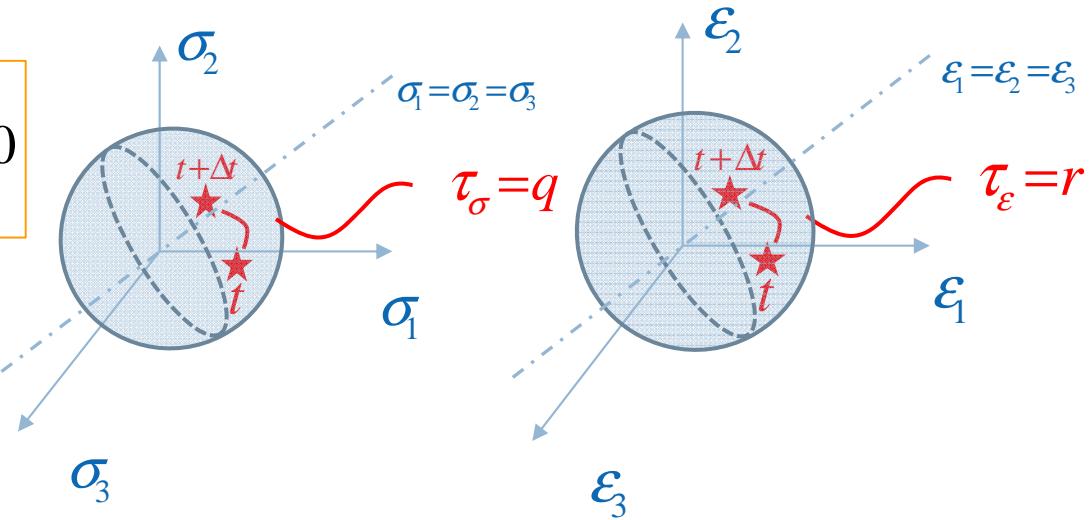
□ GEOMETRICAL POINT OF VIEW

Representative points in the stress/strain spaces

Stress/strain paths $t \rightarrow \left\{ \begin{array}{l} \sigma_t \\ \epsilon_t \end{array} \right. ; t + \Delta t \rightarrow \left\{ \begin{array}{l} \sigma_{t+\Delta t} \\ \epsilon_{t+\Delta t} \end{array} \right. ; t + 2\Delta t \rightarrow \left\{ \begin{array}{l} \sigma_{t+2\Delta t} \\ \epsilon_{t+2\Delta t} \end{array} \right. \dots$

1. While the point remains **inside** the elastic domains the internal variable does not change (neither the elastic domain)

$$\begin{cases} \sigma_t \in E_\sigma(t) \\ \sigma_{t+\Delta t} \in E_\sigma(t) = E_\sigma(t + \Delta t) \end{cases} \rightarrow \dot{r} = 0$$

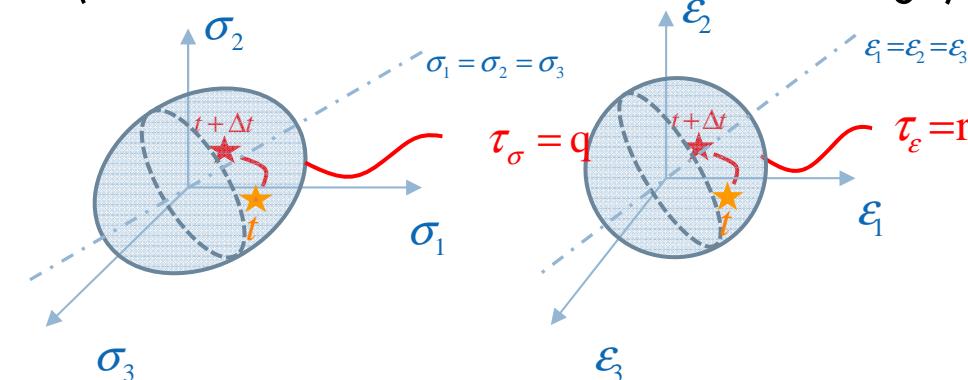


Constitutive equation-additional elements: Karush/Kuhn/Tucker conditions

2. As the point has reached **the boundary** at time $t \rightarrow (\sigma_t \in \partial E_\sigma(t))$

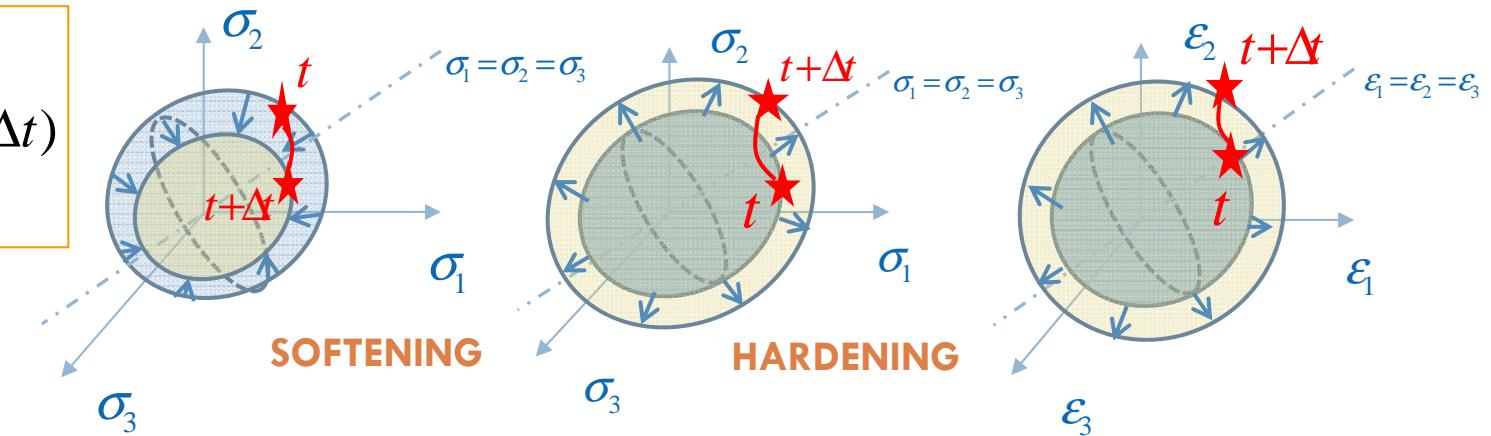
a) The point moves **inwards** (then the internal variable does not change)

$$\begin{cases} \sigma_t \in \partial E_\sigma(t) \\ \sigma_{t+\Delta t} \in E_\sigma(t) = E_\sigma(t + \Delta t) \\ \Rightarrow \dot{r} = 0 \end{cases}$$



b) The point moves **outwards** (then the internal variable changes and the elastic domains evolve)

$$\begin{cases} \sigma_t \in \partial E_\sigma(t) \\ \sigma_{t+\Delta t} \in \partial E_\sigma(t + \Delta t) \\ \Rightarrow \dot{r} \neq 0 \end{cases}$$



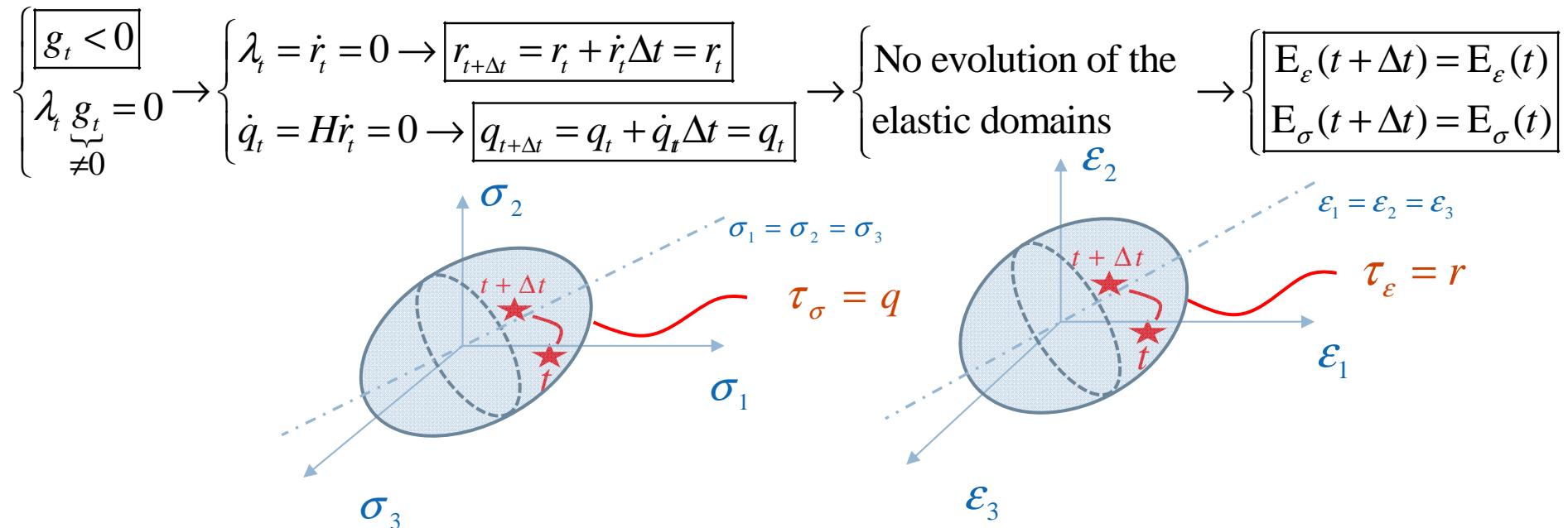
Constitutive equation-additional elements: Karush/Kuhn/Tucker conditions

□ MATHEMATICAL EXPRESSIONS OF THE K-K-T CONDITIONS

$$\lambda \geq 0 ; g \leq 0 ; \lambda g = 0 \rightarrow \text{Loading/unloading conditions}$$

for $g = 0 \quad \lambda \dot{g} = 0 \quad \rightarrow \text{Persistency/consistency conditions}$

a) Instantaneous elastic state



Constitutive equation-additional elements: Karush/Kuhn/Tucker conditions

b) Instantaneous inelastic state (damage)

$$\begin{cases} \lambda \geq 0 ; g \leq 0 ; \lambda g = 0 \\ g = 0 \quad \lambda \dot{g} = 0 \end{cases}$$

$$\sigma_t \in \partial E_\sigma(t) \rightarrow \boxed{g_t = 0} \rightarrow \underbrace{\lambda_t \dot{g}_t = 0}_{\text{persistency}}$$

$$\rightarrow \begin{cases} \dot{g}_t < 0 \rightarrow g_{t+\Delta t} = \overbrace{g_t}^{=0} + \overbrace{\dot{g}_t \Delta t}^{<0} < 0 \rightarrow \text{elastic unloading} \\ \dot{g}_t = 0 \rightarrow g_{t+\Delta t} = \overbrace{g_t}^{=0} + \overbrace{\dot{g}_t \Delta t}^{=0} = 0 \rightarrow \text{loading} \\ \dot{g}_t > 0 \rightarrow g_{t+\Delta t} = \overbrace{g_t}^{=0} + \overbrace{\dot{g}_t \Delta t}^{>0} > 0 \rightarrow \text{unfeasible} \end{cases}$$

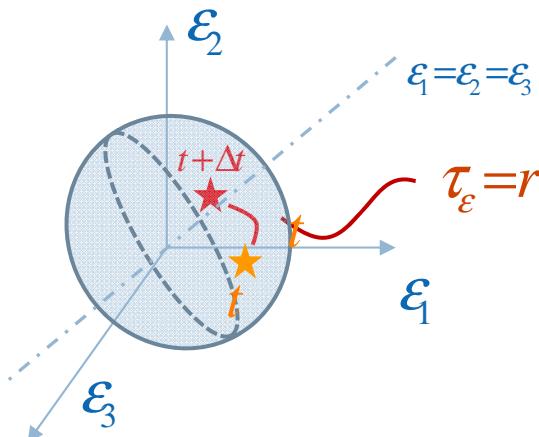
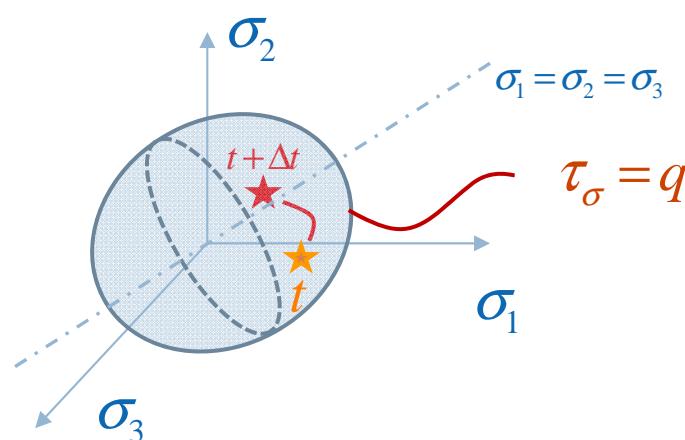
Constitutive equation-additional elements: Karush/Kuhn/Tucker conditions

1. Elastic unloading

$$\begin{cases} \lambda \geq 0 ; g \leq 0 ; \lambda g = 0 \\ g = 0 \quad \lambda \dot{g} = 0 \end{cases}$$

$$\boxed{\dot{g}_t < 0} \rightarrow \begin{cases} g_t = 0 \\ g_{t+\Delta t} = g_t + \underbrace{\dot{g}_t}_{\leq 0} \Delta t < 0 \end{cases} \rightarrow \begin{cases} \text{Point evolves towards} \\ \text{the interior of } E_\varepsilon(t + \Delta t) \end{cases}$$

$$\lambda_t \underbrace{\dot{g}_t}_{\neq 0} = 0 \rightarrow \begin{cases} \lambda_t = \dot{r}_t = 0 \rightarrow \boxed{r_{t+\Delta t} = r_t + \dot{r}_t \Delta t = r_t} \\ \dot{q}_t = H \dot{r}_t = 0 \rightarrow \boxed{q_{t+\Delta t} = q_t + \dot{q}_t \Delta t = q_t} \end{cases} \rightarrow \begin{cases} \text{No evolution of the} \\ \text{elastic domains} \end{cases} \rightarrow \begin{cases} \boxed{E_\varepsilon(t + \Delta t) = E_\varepsilon(t)} \\ \boxed{E_\sigma(t + \Delta t) = E_\sigma(t)} \end{cases}$$



Constitutive equation-additional elements:

Karush/Kuhn/Tucker conditions

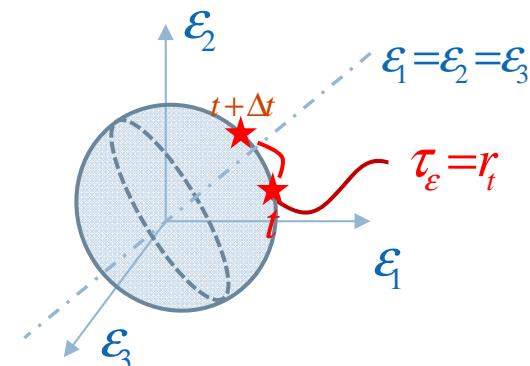
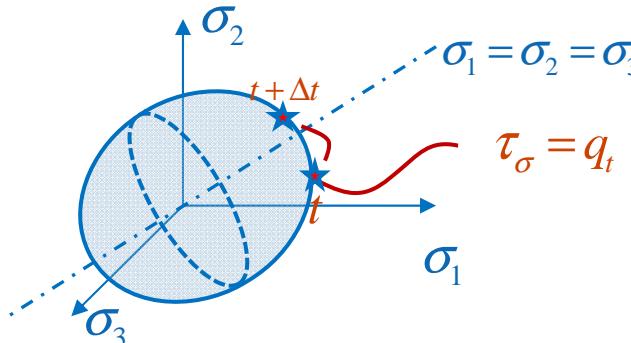
$$\begin{cases} \lambda \geq 0 ; g \leq 0 ; \lambda g = 0 \\ g = 0 \quad \lambda \dot{g} = 0 \end{cases}$$

2. Loading

$$\boxed{\dot{g}_t = 0} \rightarrow \begin{cases} g_t = 0 \\ g_{t+\Delta t} = g_t + \underbrace{\dot{g}_t}_{=0} \Delta t = 0 \end{cases} \rightarrow \begin{cases} \text{Point remains on the} \\ \text{boundary } \partial E_\varepsilon(t + \Delta t) \end{cases} \quad \lambda_t \underbrace{\dot{g}_t}_{=0} = 0 \rightarrow \begin{cases} \lambda_t = 0 \rightarrow \text{Neutral loading} \\ \lambda_t > 0 \rightarrow \text{Pure loading} \end{cases}$$

■ Neutral loading

$$\begin{cases} \lambda_t = \dot{r}_t = 0 \rightarrow \boxed{r_{t+\Delta t} = r_t + \dot{r}_t \Delta t = r_t} \\ \dot{q}_t = H \dot{r}_t = 0 \rightarrow \boxed{q_{t+\Delta t} = q_t + \dot{q}_t \Delta t = q_t} \end{cases} \rightarrow \begin{cases} \text{No evolution of the} \\ \text{elastic domains} \end{cases} \rightarrow \begin{cases} \boxed{E_\varepsilon(t + \Delta t) = E_\varepsilon(t)} \\ \boxed{E_\sigma(t + \Delta t) = E_\sigma(t)} \end{cases}$$



Remark

Point moves on the boundary of the stationary domains $E_\sigma(t), E_\varepsilon(t)$

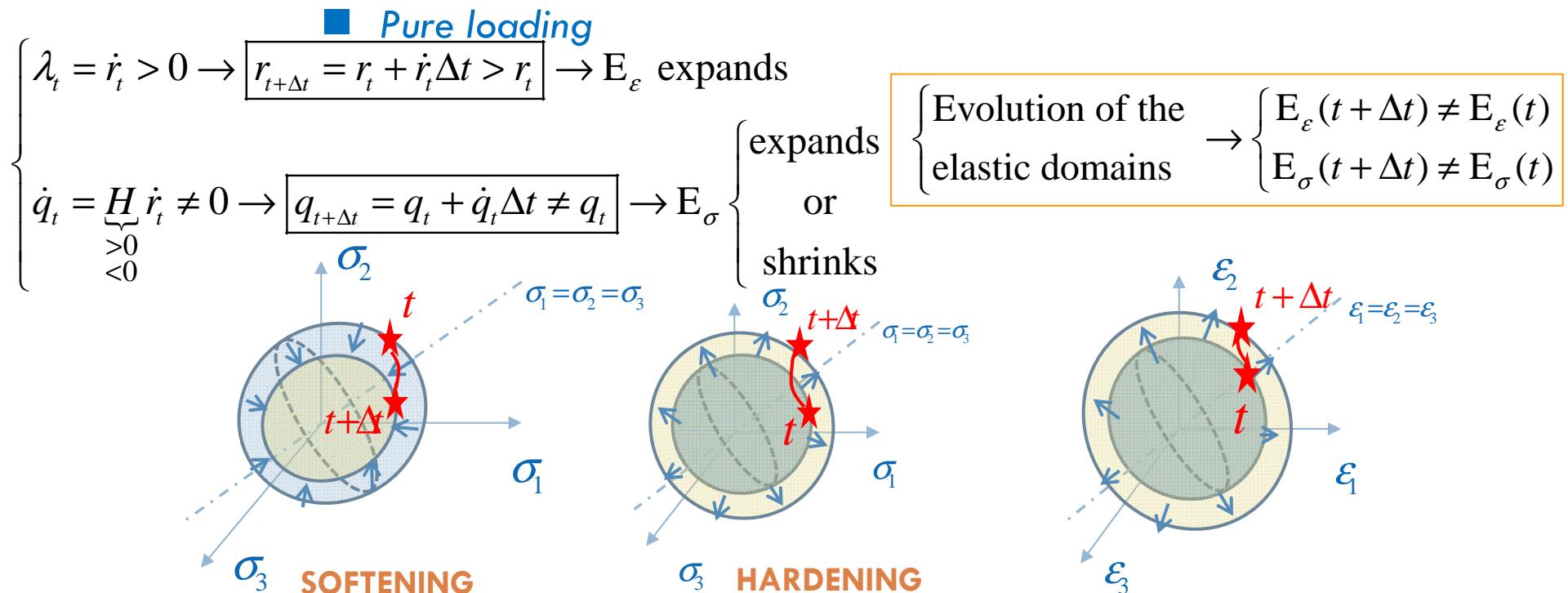
Constitutive equation-additional elements:

Karush/Kuhn/Tucker conditions

$$\begin{cases} \lambda \geq 0 ; g \leq 0 ; \lambda g = 0 \\ g = 0 \quad \lambda \dot{g} = 0 \end{cases}$$

2. Loading

$$\boxed{\dot{g}_t = 0} \rightarrow \begin{cases} g_t = 0 \\ g_{t+\Delta t} = g_t + \underbrace{\dot{g}_t}_{=0} \Delta t = 0 \end{cases} \rightarrow \begin{cases} \text{Point remains on the} \\ \text{boundary } \partial E_\varepsilon(t + \Delta t) \end{cases} \quad \lambda_t \underbrace{\dot{g}_t}_{\equiv 0} = 0 \rightarrow \begin{cases} \lambda_t = 0 \rightarrow \text{Neutral loading} \\ \lambda_t > 0 \rightarrow \text{Pure loading} \end{cases}$$



Isotropic continuum damage model (Summary)

- Free energy

$$\psi(\boldsymbol{\varepsilon}, d(r)) = (1 - d(r))\psi_0(\boldsymbol{\varepsilon}) = (1 - d(r)) \frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})$$

- Damage variable

$$d := 1 - \frac{q(r)}{r} \quad 0 \leq d \leq 1 \quad q(r_0) = r_0 \quad \text{for } d = 0$$

- Constitutive equation

$$\sigma = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \frac{\partial \psi_0(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \mathbb{C} : \boldsymbol{\varepsilon}$$

Isotropic continuum damage model (Summary)

□ Evolution equation

$$\dot{r} = \lambda(\boldsymbol{\varepsilon}, r) \geq 0, \quad r|_{t=0} = r_0, \quad r \in [r_0, \infty]$$

□ Damage criterion

$$f(\boldsymbol{\sigma}, r) = \tau_\sigma - q(r) \leq 0 \quad \longrightarrow \quad \tau_\sigma = \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} = \sqrt{2G(\boldsymbol{\sigma})}$$

$$g(\boldsymbol{\varepsilon}, r) = \tau_\varepsilon - r \leq 0 \quad \longrightarrow \quad \tau_\varepsilon = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}} = \sqrt{2\psi_0(\boldsymbol{\varepsilon})}$$

□ Loading/unloading conditions

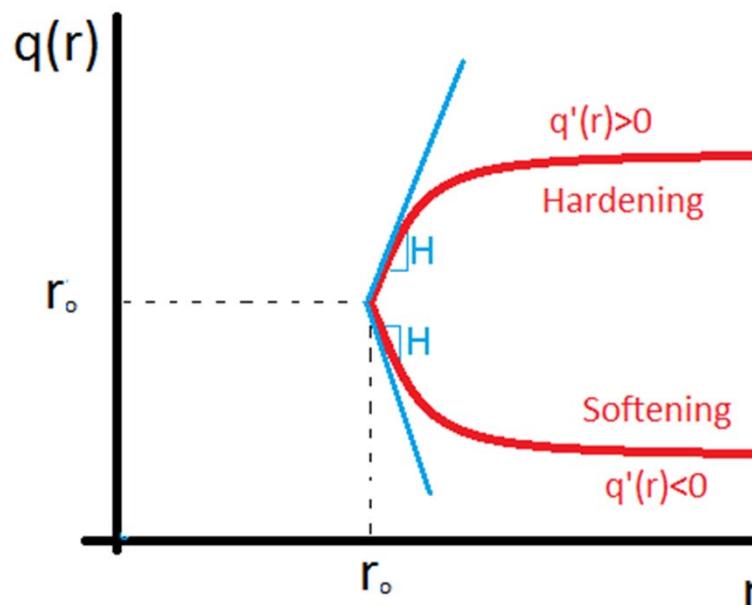
$\lambda \geq 0 ; g \leq 0 ; \lambda g = 0 \rightarrow$ Loading/unloading conditions

for $g = 0 \quad \lambda \dot{g} = 0 \quad \rightarrow$ Persistency/consistency conditions

Isotropic continuum damage model (Summary)

□ Hardening/softening law

$$H := \frac{dq(r)}{dr} = q'(r) \quad \begin{array}{c} H < 0 \\ \text{Softening} \end{array} \quad \begin{array}{c} H > 0 \\ \text{Hardening} \end{array}$$





END OF LECTURE 3