

Integration of the constitutive equation

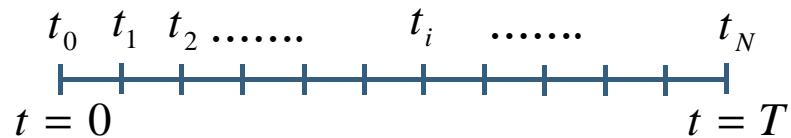
REMAINDER ON NUMERICAL INTEGRATION

- Analytical integration

$$\left. \begin{array}{l} f(x(t), \dot{x}(t)) = 0 \\ x|_{t=0} = x_0 \end{array} \right\} \rightarrow x = f(t)$$

Exact/closed-form solution (not always possible)

- Numerical integration



$$\left\{ \begin{array}{l} [0, T] = \bigcup_{i=1}^{i=N} [t_i, t_{i+1}] \\ [t_i, t_{i+1}] \rightarrow \text{Time interval/step "i+1"} \\ \Delta t = t_{i+1} - t_i \rightarrow \text{Time increment} \end{array} \right.$$

$$x(t_{i+1}) \equiv x_{i+1} = g(x_i, x_{i-1}, x_{i-2}, \dots)$$

Approximate solution (depending on the values at previous discrete times)

Integration of the evolution equation

Loading/Unloading equations

$$\begin{cases} \lambda_t \geq 0 ; g_t \leq 0 ; \lambda_t g_t = 0 \\ g_t = 0 \quad \lambda_t \dot{g}_t = 0 \end{cases} \rightarrow \begin{cases} \dot{r}_t = \lambda_t(\varepsilon_t, r_t) \\ g_t \equiv \boxed{\tau_\varepsilon(t) - r_t \leq 0} \end{cases} \xrightarrow{\text{Time integration}} r_t = f(\varepsilon_t)$$

a) ELASTIC STATE

$$\begin{cases} g_t \equiv \tau_\varepsilon - r_t < 0 \\ \boxed{\tau_\varepsilon(t) < r_t} \end{cases} \rightarrow \lambda_t = \dot{r}_t = 0 \rightarrow \text{No evolution of } r_t \rightarrow \boxed{\dot{r}_t = 0 \rightarrow \begin{cases} \dot{\tau}_\varepsilon(t) < 0 \\ \dot{\tau}_\varepsilon(t) \geq 0 \end{cases}}$$

b) INELASTIC (DAMAGE) STATE

$\begin{cases} g_t \equiv \tau_\varepsilon - r_t = 0 \\ \boxed{\tau_\varepsilon(t) = r_t} \end{cases}$	b-1) UNLOADING $\dot{g}_t \equiv \dot{\tau}_\varepsilon(t) - \dot{r}_t < 0 \rightarrow \lambda_t = \dot{r}_t = 0 \rightarrow \text{No evolution of } r_t \rightarrow \boxed{\dot{r}_t = 0 \rightarrow \dot{\tau}_\varepsilon(t) < 0}$
	b-2-1) NEUTRAL LOADING $\lambda_t = \dot{r}_t = 0 \rightarrow \text{No evolution of } r_t \rightarrow \boxed{\dot{r}_t = 0 \rightarrow \dot{\tau}_\varepsilon(t) = 0}$
	b-2) LOADING $\dot{g}_t \equiv \dot{\tau}_\varepsilon(t) - \dot{r}_t = 0 \rightarrow \begin{cases} \text{b-2-2) PURE LOADING} \\ \lambda_t = \dot{r}_t = \dot{\tau}_\varepsilon(t) > 0 \rightarrow \text{Evolution of } r_t \rightarrow \boxed{\dot{r}_t = \dot{\tau}_\varepsilon(t) \geq 0} \end{cases}$

Integration of the evolution and constitutive equations

SUMMARY :

- 1) the initial value of r is $r_0 > 0$
- 2) the initial value of τ_ε is $\tau_\varepsilon(0) = 0$
- 3) r never decreases ($\dot{r} \geq 0$)
- 4) τ_ε is always smaller or equal than r ($g \equiv \tau_\varepsilon - r \leq 0$)
- 5) When τ_ε equals r then

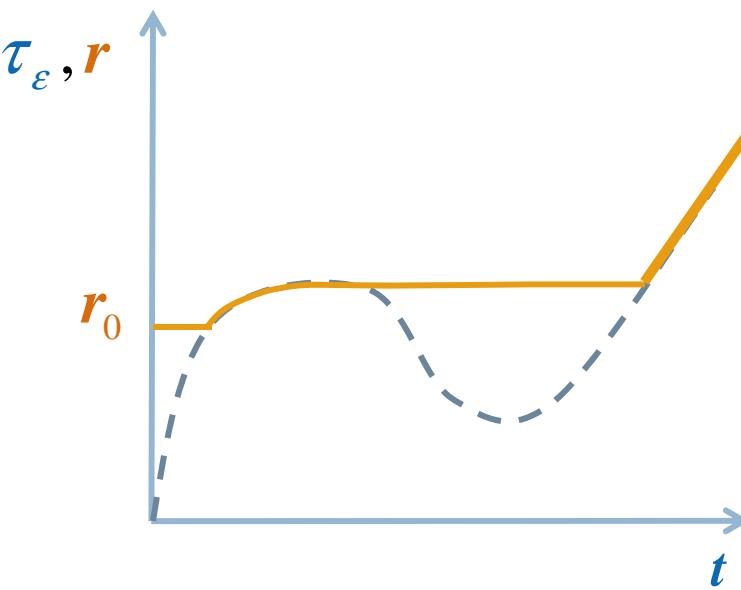
if τ_ε grows $\rightarrow \dot{\tau}_\varepsilon = \dot{r} \geq 0$
if τ_ε decreases $\rightarrow \dot{\tau}_\varepsilon < 0 ; \dot{r} = 0$

$$r(t) = \underbrace{\max(r_0, \tau_\varepsilon(s))}_{\text{historical maximum of } [r_0, \tau_\varepsilon(s)]} \quad s \in [0, t]$$

$$\boldsymbol{\varepsilon}_s \rightarrow \tau_\varepsilon(s) \rightarrow r_t \rightarrow q_t = q(r_t) \rightarrow$$

$$d_t = 1 - \frac{q_t}{r_t} \rightarrow \boldsymbol{\sigma}_t = (1 - d_t) \mathbb{C} : \boldsymbol{\varepsilon}_t$$

The integration is exact (closed form).
No dependence on Δt

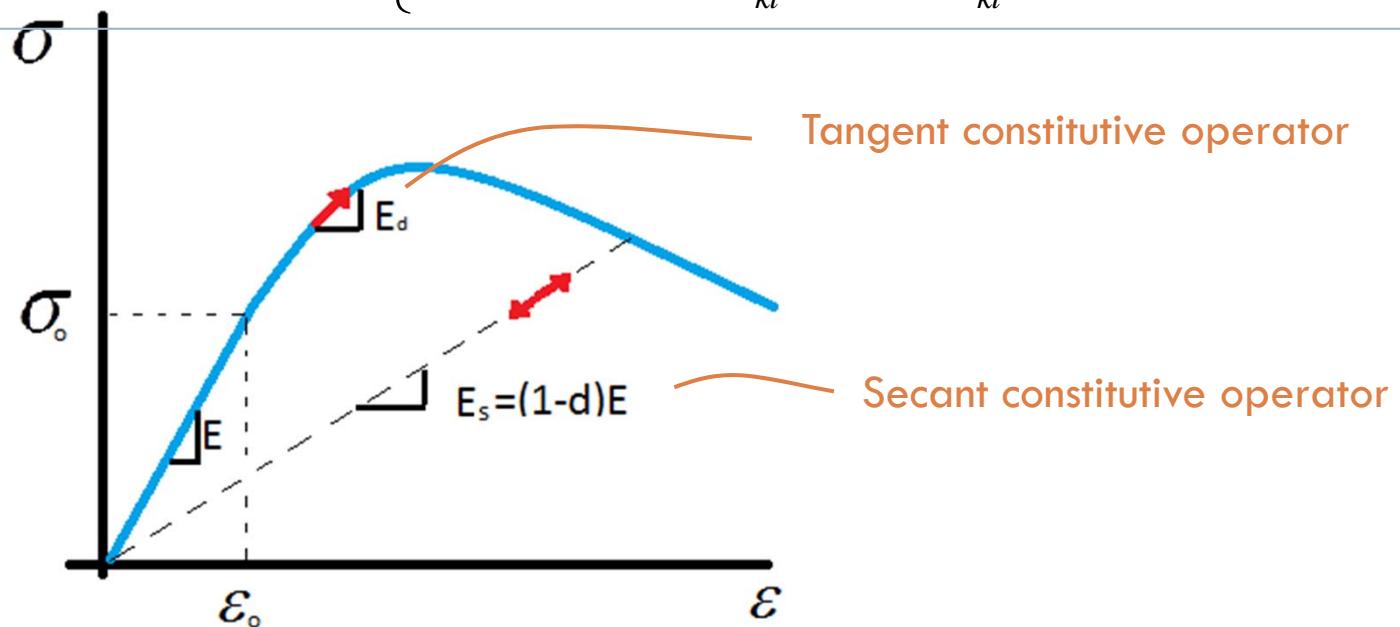


Tangent constitutive operator (definition)

$$\boldsymbol{\varepsilon}(\mathbf{x}, t) \rightarrow \boldsymbol{\sigma}(\mathbf{x}, t) = \Sigma(\boldsymbol{\varepsilon}(\mathbf{x}, t))$$

Tangent constitutive operator \rightarrow

$$\begin{cases} \mathbb{C}_{\text{tang}}^d(\boldsymbol{\varepsilon}) = \frac{\partial \Sigma(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \equiv \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} \\ [\mathbb{C}_{\text{tang}}^d]_{ijkl} = \frac{\partial \Sigma_{ij}(\boldsymbol{\varepsilon})}{\partial \varepsilon_{kl}} \equiv \frac{\partial \sigma_{ij}(\boldsymbol{\varepsilon})}{\partial \varepsilon_{kl}} \quad i, j, k, l \in \{1, 2, 3\} \end{cases}$$



Tangent constitutive operator (analytical derivation)

- Tangent constitutive moduli

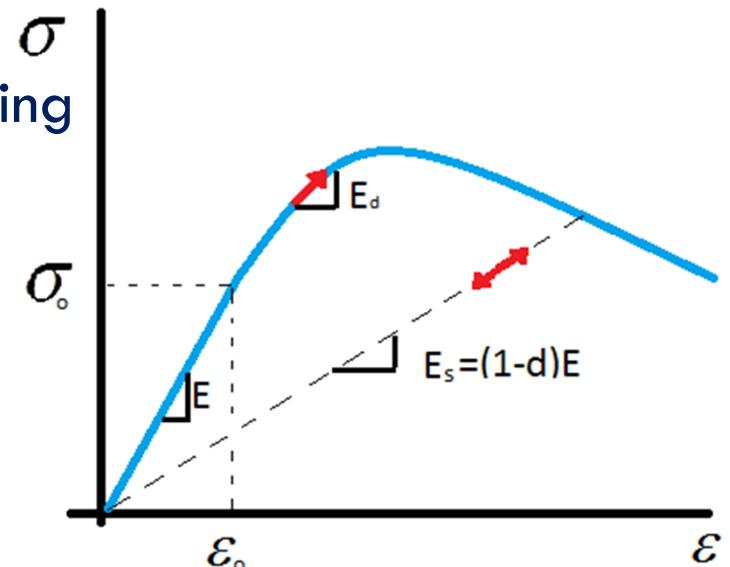
$$\sigma(\varepsilon(t), \dot{\varepsilon}) = (1-d(\varepsilon))\mathbb{C}(\varepsilon) : \varepsilon \quad \rightarrow \begin{cases} d\sigma(\varepsilon(t)) = \mathbb{C}_{\text{tang}}^d(\varepsilon) : d\varepsilon \\ \dot{\sigma}(\varepsilon) = \frac{d\sigma}{dt} = \frac{\partial\sigma(\varepsilon)}{\partial\varepsilon} : \frac{d\varepsilon}{dt} = \mathbb{C}_{\text{tang}}^d : \frac{d\varepsilon}{dt} \end{cases} \Rightarrow \boxed{\dot{\sigma}_t = \mathbb{C}_{\text{tang}}^d(t) : \dot{\varepsilon}_t}$$

- Derivation $\dot{\sigma} = (1-d)\mathbb{C} : \dot{\varepsilon} - d\mathbb{C} : \varepsilon = \mathbb{C}_{\text{tang}}^d : \dot{\varepsilon}$

- Elastic regime/Unloading/Neutral loading

$$\dot{r} = 0 \rightarrow \dot{d} = 0 \rightarrow \dot{\sigma} = \underbrace{(1-d)\mathbb{C}}_{\mathbb{C}_{\text{tang}}^d} : \dot{\varepsilon}$$

$$\mathbb{C}_{\text{tang}}^d = (1-d)\mathbb{C} = \mathbb{C}_{\text{sec}}^d \quad (\text{secant constitutive operator})$$



Tangent constitutive operator (analytical derivation)

□ Inelastic regime-loading

$$\begin{cases} \tau_\varepsilon(t) = r_t \\ \dot{\tau}_\varepsilon = \dot{r}_t > 0 \end{cases} \quad \begin{cases} d(r) = 1 - \frac{q(r)}{r} \\ \dot{d}(t) = d'(r)\dot{r} = \frac{-q'(r)r + q(r)}{r^2}\dot{r} \end{cases} \rightarrow \boxed{\dot{d}(t) = \frac{q - Hr}{r^3}\dot{r} \geq 0}$$

$$\begin{cases} r = \tau_\varepsilon = \sqrt{\bar{\sigma} : \mathbb{M} : \bar{\sigma}} \\ \dot{r} = \dot{\tau}_\varepsilon \end{cases} \rightarrow \dot{r} = \frac{d}{dt} \sqrt{\bar{\sigma} : \mathbb{M} : \bar{\sigma}} = \frac{1}{2\sqrt{\bar{\sigma} : \mathbb{M} : \bar{\sigma}}} \underbrace{\frac{d}{dt}(\bar{\sigma} : \mathbb{M} : \bar{\sigma})}_{\tau_\varepsilon = r} = \frac{1}{2\bar{\sigma} : \mathbb{M} : \dot{\bar{\sigma}}}$$

$$= \frac{1}{2r} \cancel{\cancel{\bar{\sigma} : \mathbb{M} : \dot{\bar{\sigma}}}} = \frac{1}{r} \bar{\sigma} : \underbrace{\mathbb{M} : \mathbb{C}}_{\equiv \mathbb{A}} : \dot{\varepsilon} = \frac{1}{r} \bar{\sigma} : \mathbb{A} : \dot{\varepsilon} ; \quad \boxed{\dot{r} = \frac{1}{r} \bar{\sigma} : \mathbb{A} : \dot{\varepsilon}}$$

$$\begin{cases} \dot{d}(t) = \frac{q - Hr}{r^3}\dot{r} \\ \dot{r} = \frac{1}{r} \bar{\sigma} : \mathbb{A} : \dot{\varepsilon} \end{cases} \rightarrow \boxed{\dot{d} = \frac{q - Hr}{r^3} \bar{\sigma} : \mathbb{A} : \dot{\varepsilon}}$$

Tangent constitutive operator (analytical derivation)

$$\dot{\sigma} = (1-d)\mathbb{C}:\dot{\epsilon} - \underbrace{d\mathbb{C}:\bar{\sigma}}_{\bar{\sigma}} = (1-d)\mathbb{C}:\dot{\epsilon} - \underbrace{d\dot{\bar{\sigma}}}_{\bar{\sigma}\otimes d} \quad ; \quad d = -\frac{Hr-q}{r^3} \bar{\sigma}:\mathbb{A}:\dot{\epsilon}$$

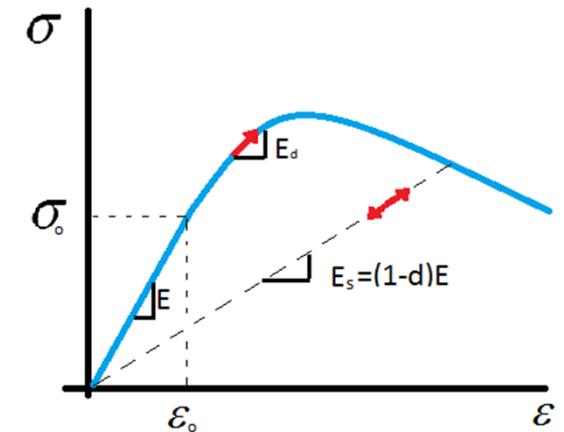
$$\dot{\sigma} = (1-d)\mathbb{C}:\dot{\epsilon} - \frac{q-Hr}{r^3} \bar{\sigma} \otimes (\bar{\sigma}:\mathbb{A}):\dot{\epsilon} = \underbrace{[(1-d)\mathbb{C} - \frac{q-Hr}{r^3} (\bar{\sigma} \otimes [\bar{\sigma}:\mathbb{A}])]}_{\mathbb{C}_{\text{tang}}^d} : \dot{\epsilon} = \mathbb{C}_{\text{tang}}^d : \dot{\epsilon}$$

$$\mathbb{C}_{\text{tang}}^d(\boldsymbol{\epsilon}) = (1-d)\mathbb{C} - \frac{q-Hr}{r^3} (\bar{\sigma} \otimes [\bar{\sigma}:\mathbb{A}])$$

Symmetric model $\rightarrow \begin{cases} \mathbb{M} = \mathbb{C}^{-1} \\ \mathbb{A} = \mathbb{M} : \mathbb{C} \end{cases} \rightarrow \begin{cases} \mathbb{A} = \mathbb{C}^{-1} : \mathbb{C} = \mathbb{I} \\ \bar{\sigma} : \mathbb{A} = \bar{\sigma} : \mathbb{I} = \bar{\sigma} \end{cases}$

(tension/compression) $\rightarrow \begin{cases} (1-d)\mathbb{C} \rightarrow (\text{elastic/unloading}) \\ (1-d)\mathbb{C} - \frac{q-Hr}{r^3} (\bar{\sigma} \otimes \bar{\sigma}) \rightarrow (\text{loading}) \end{cases}$

$$\mathbb{C}_{\text{tang}}^d(\boldsymbol{\epsilon}) = \begin{cases} (1-d)\mathbb{C} \rightarrow (\text{elastic/unloading}) \\ (1-d)\mathbb{C} - \frac{q-Hr}{r^3} (\bar{\sigma} \otimes \bar{\sigma}) \rightarrow (\text{loading}) \end{cases}$$



Integration of the constitutive equation: Identification of the current state

Identification at the end of the current time interval) $[t, t + \Delta t]$

1) Elastic

$$\begin{cases} g_{t+\Delta t} \equiv \tau_{\varepsilon_{t+\Delta t}} - r_{t+\Delta t} < 0 \\ \dot{r}_{t+\Delta t} = \frac{r_{t+\Delta t} - r_t}{\Delta t} = 0 \rightarrow r_{t+\Delta t} = r_t \end{cases} \rightarrow \tau_{\varepsilon_{t+\Delta t}} - r_t < 0 \rightarrow \boxed{\tau_{\varepsilon_{t+\Delta t}} < r_t}$$

2) Unloading/Neutral loading

$$\begin{cases} g_{t+\Delta t} \equiv \tau_{\varepsilon_{t+\Delta t}} - r_{t+\Delta t} = 0 \\ \dot{r}_{t+\Delta t} = \frac{r_{t+\Delta t} - r_t}{\Delta t} = 0 \rightarrow r_{t+\Delta t} = r_t \end{cases} \rightarrow \tau_{\varepsilon_{t+\Delta t}} - r_t = 0 \rightarrow \boxed{\tau_{\varepsilon_{t+\Delta t}} = r_t}$$

3) Loading

$$\begin{cases} g_{t+\Delta t} \equiv \tau_{\varepsilon_{t+\Delta t}} - r_{t+\Delta t} = 0 \\ \dot{r}_{t+\Delta t} = \frac{r_{t+\Delta t} - r_t}{\Delta t} > 0 \rightarrow r_{t+\Delta t} > r_t \end{cases} \rightarrow \begin{cases} r_{t+\Delta t} > r_t \\ \tau_{\varepsilon_{t+\Delta t}} = r_{t+\Delta t} > r_t \end{cases} \rightarrow \boxed{\tau_{\varepsilon_{t+\Delta t}} > r_t}$$

Integration of the constitutive equation

Numerical algorithm:

INPUT DATA $[t, t + \Delta t] \rightarrow \boldsymbol{\varepsilon}_t, r_t, \boldsymbol{\varepsilon}_{t+\Delta t}$

Step 1 → Compute $\begin{cases} \bar{\boldsymbol{\sigma}}_{t+\Delta t} = \mathbb{C} : \boldsymbol{\varepsilon}_{t+\Delta t} \\ \tau_{\boldsymbol{\varepsilon}_{t+\Delta t}} = \sqrt{\boldsymbol{\varepsilon}_{t+\Delta t} : \mathbb{C} : \boldsymbol{\varepsilon}_{t+\Delta t}} = \sqrt{\boldsymbol{\varepsilon}_{t+\Delta t} : \bar{\boldsymbol{\sigma}}_{t+\Delta t}} \end{cases}$



Step 2 → If $\tau_{\boldsymbol{\varepsilon}_{t+\Delta t}} \leq r_t \rightarrow$

Elastic
Unloading
Neutral loading

$\rightarrow \begin{cases} r_{t+\Delta t} = r_t \\ d_{t+\Delta t} = d_t = 1 - \frac{q(r_{t+\Delta t})}{r_{t+\Delta t}} \\ \boldsymbol{\sigma}_{t+\Delta t} = (1 - d_{t+\Delta t}) \bar{\boldsymbol{\sigma}}_{t+\Delta t} \\ (\mathbb{C}_{\text{tang}}^d)_{t+\Delta t} = (1 - d_{t+\Delta t}) \mathbb{C} \end{cases} \rightarrow \text{EXIT}$



Integration of the constitutive equation

Numerical algorithm:



Step 3 → If $\tau_{\varepsilon_{t+\Delta t}} > r_t \rightarrow$ (Loading)

$$\rightarrow \begin{cases} r_{t+\Delta t} = \tau_{\varepsilon_{t+\Delta t}} \\ d_{t+\Delta t} = 1 - \frac{q(r_{t+\Delta t})}{r_{t+\Delta t}} \\ \boldsymbol{\sigma}_{t+\Delta t} = (1 - d_{t+\Delta t}) \bar{\boldsymbol{\sigma}}_{t+\Delta t} \\ (\mathbb{C}_{\text{tang}}^d)_{t+\Delta t} = (1 - d_{t+\Delta t}) \mathbb{C} - \frac{q(r_{t+\Delta t}) - H_{t+\Delta t} r_{t+\Delta t}}{(r_{t+\Delta t})^3} (\bar{\boldsymbol{\sigma}}_{t+\Delta t} \otimes \bar{\boldsymbol{\sigma}}_{t+\Delta t}) \end{cases}$$

→ EXIT

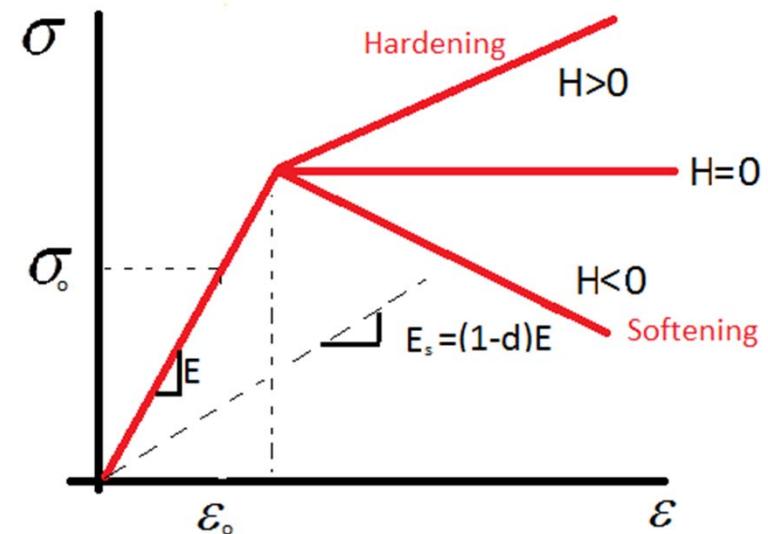
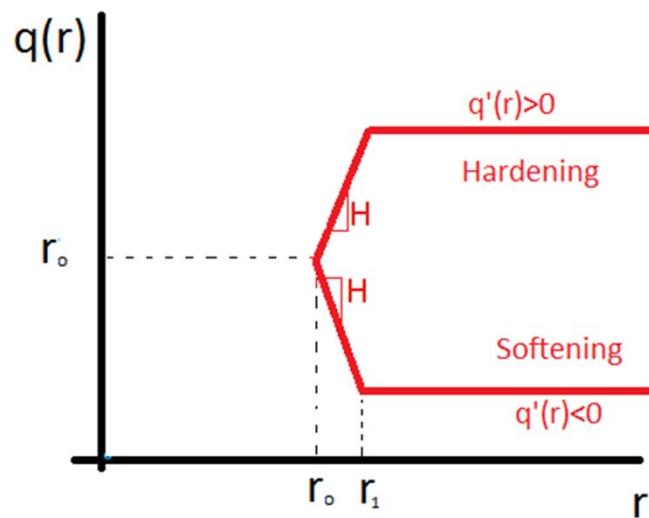
OUTPUT DATA $[t, t + \Delta t] \rightarrow r_{t+\Delta t}, \boldsymbol{\sigma}_{t+\Delta t}, (\mathbb{C}_{\text{tang}}^d)_{t+\Delta t}$

Characterization of the Hardening/Softening law

- Linear hardening/softening

$$q(r) = q(r) = \begin{cases} r_0 + H(r - r_0) & r \in [r_0, r_1] \quad (r_1 = r_0 + \frac{1}{H}(q_\infty - r_0)) \\ q_\infty > 0 & r \in [r_1, \infty) \end{cases}$$

$$H = \frac{dq(r)}{dr} = \begin{cases} H & r \in [r_0, r_1) \\ 0 & r \in [r_1, \infty) \end{cases}$$

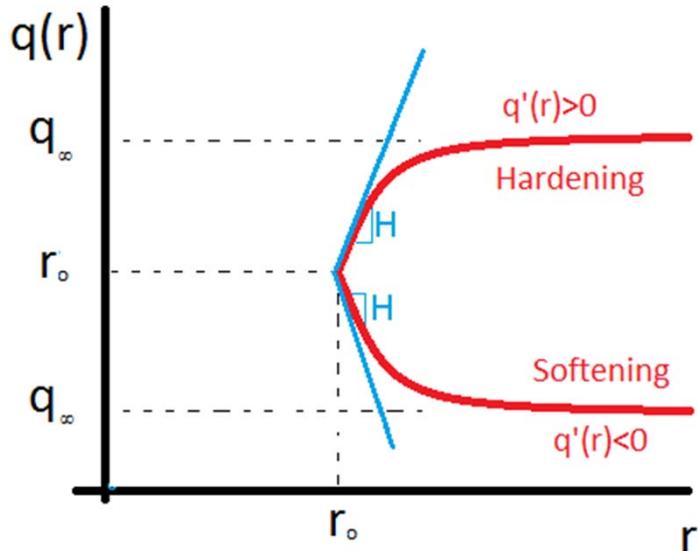


Characterization of the Hardening/Softening law

- Exponential hardening/softening

$$q(r) = q_\infty - (q_\infty - r_0) e^{-A(1-\frac{r}{r_0})} \quad (A > 0)$$

$$H(r) = \frac{dq(r)}{dr} = A \frac{(q_\infty - r_0)}{r_0} e^{-A(1-\frac{r}{r_0})}$$



Characterization of the Hardening/Softening law

- Value of r_0

It is computed from the 1D case



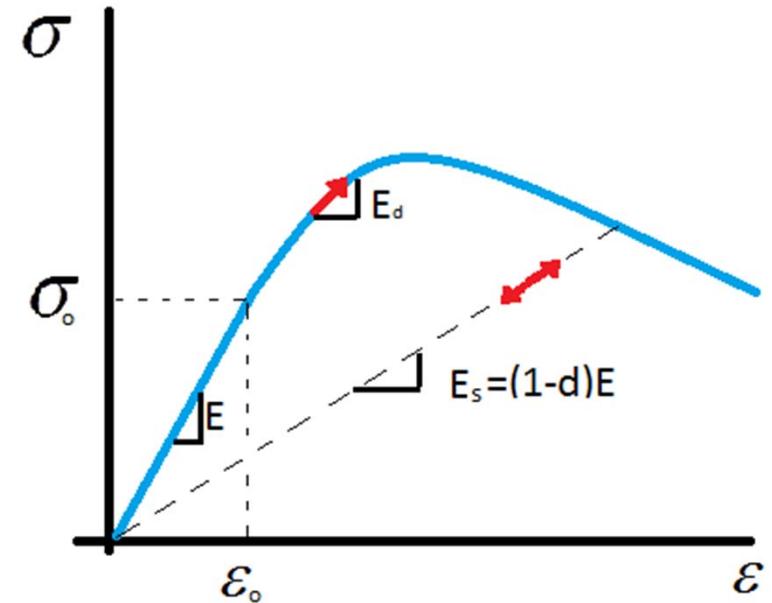
$$r = \tau_\varepsilon = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}} = \sqrt{\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}}$$

$$\bar{\boldsymbol{\sigma}}_u = \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \boldsymbol{\varepsilon}_u = \begin{bmatrix} \frac{\sigma_u}{E} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{\sigma}_u : \boldsymbol{\varepsilon}_u = \sigma_{ij} \varepsilon_{ij} = \frac{(\sigma_u)^2}{E}$$

$$r_0 = \sqrt{\bar{\boldsymbol{\sigma}}_u : \boldsymbol{\varepsilon}_u} = \frac{\sigma_u}{\sqrt{E}}$$

$$\boldsymbol{\sigma}_u = \bar{\boldsymbol{\sigma}}_u = \begin{bmatrix} \sigma_u & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \sigma_u = \text{uniaxial elastic limit}$$



Characterization of the elastic domain

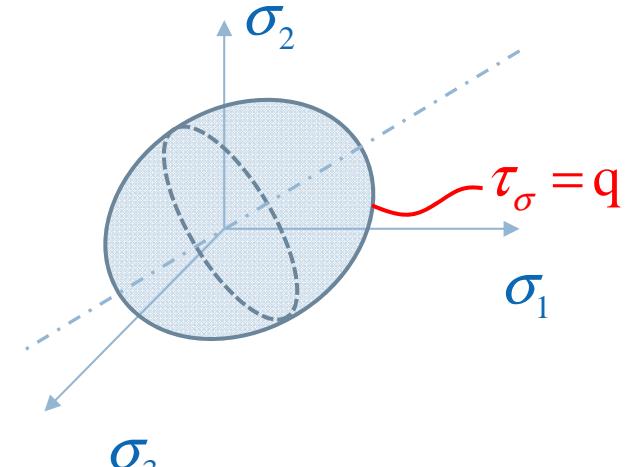
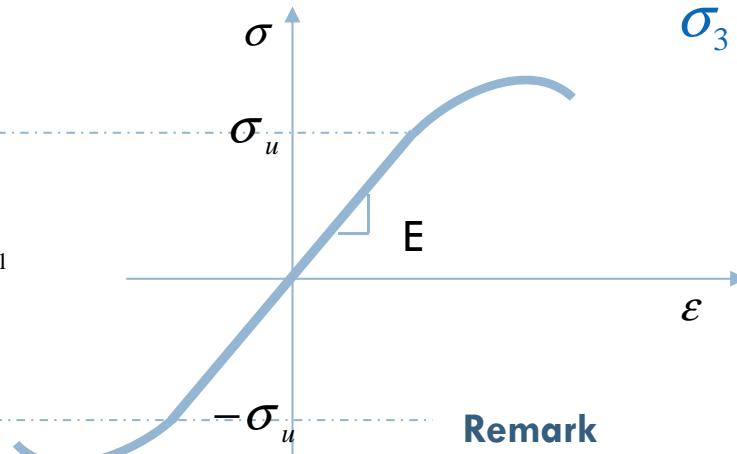
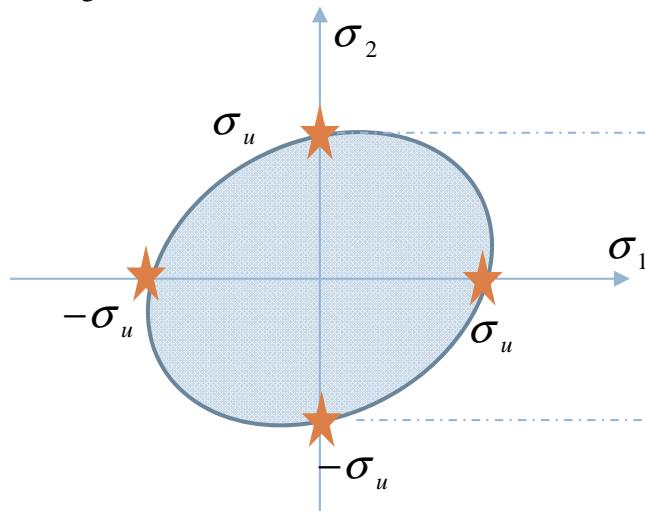
Elastic domain $\rightarrow E_\sigma := \{\sigma \in \mathbb{S} \mid f(\sigma, r) = \tau_\sigma - q(r) \leq 0\}$

1. Symmetric (tension/compression) model

$$\mathbb{M} = \mathbb{C}^{-1}$$

$$\tau_\sigma = \sqrt{\sigma : \mathbb{C}^{-1} : \sigma} = (1-d)\tau_\varepsilon = (1-d)\sqrt{\varepsilon : \mathbb{C} : \varepsilon}$$

$$\tau_\varepsilon = \sqrt{\varepsilon : \mathbb{C} : \varepsilon}$$



Remark
Equal tension and compression
elastic limits.

Characterization of the elastic domain

1. Tensile-damage-only model

- Positive counterpart of a scalar function (McAuley bracket)

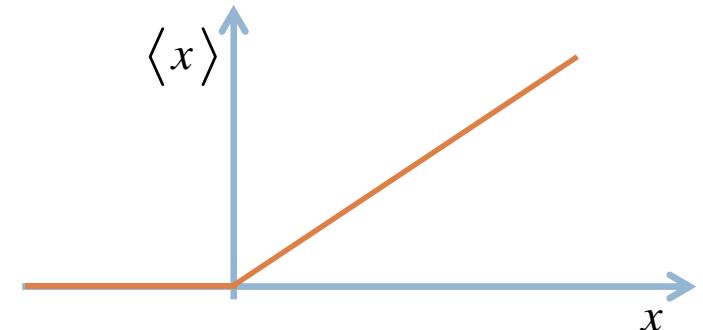
$$\langle x \rangle = \begin{cases} x & x > 0 \\ 0 & x < 0 \end{cases}$$

- Positive counterpart of a stress tensor

In matrix format:

$$[\sigma] \rightarrow \text{diagonalization} \rightarrow [\sigma]_{diag} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

$$[\sigma]_{diag}^+ = \begin{bmatrix} \langle \sigma_1 \rangle & 0 & 0 \\ 0 & \langle \sigma_2 \rangle & 0 \\ 0 & 0 & \langle \sigma_3 \rangle \end{bmatrix} \rightarrow \begin{array}{l} \text{return to original} \\ \text{system of} \\ \text{coordinates} \end{array} \rightarrow \underbrace{[\sigma]^+}_{\substack{\text{Positive} \\ \text{counterpart} \\ \text{of } [\sigma]}}$$



Characterization of the elastic domain

In tensor format:

$$\left. \begin{array}{l} \sigma = \sum_{i=1}^{i=3} \underbrace{\sigma_i}_{\text{eigenvalue "i"}} \overbrace{\hat{\mathbf{p}}_i \otimes \hat{\mathbf{p}}_i}^{\text{eigenvector "i"}} \\ \sigma^+ = \sum_{i=1}^{i=3} \langle \sigma_i \rangle \hat{\mathbf{p}}_i \otimes \hat{\mathbf{p}}_i \end{array} \right\} \rightarrow \begin{cases} \sigma \text{ and } \sigma^+ \text{ have the same eigenvectors} \\ \sigma^+ \text{ shares the positive eigenvalues of } \sigma \\ \sigma^+ \text{ has those negative eigenvalues of } \sigma \text{ null} \end{cases}$$

$$\sigma = \underbrace{(1-d)}_{\geq 0} \bar{\sigma} \Rightarrow \sigma^+ = (1-d) \bar{\sigma}^+$$

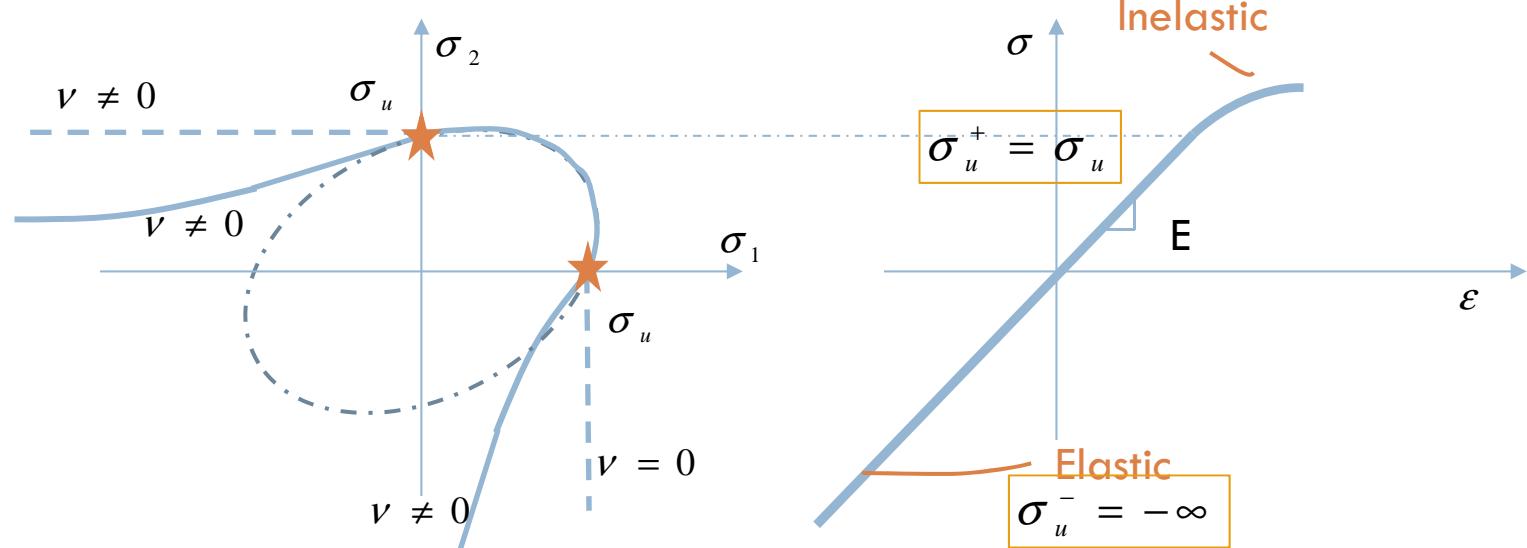
□ Strain norm redefinition

$$\tau_\sigma^+ \equiv \sqrt{\sigma^+ : \mathbb{C}^{-1} : \sigma} = \sqrt{(1-d)^2 \bar{\sigma}^+ : \mathbb{C}^{-1} : \bar{\sigma}} = (1-d) \overbrace{\sqrt{\bar{\sigma}^+ : \mathbb{C}^{-1} : \bar{\sigma}}}^{\equiv \tau_\varepsilon^+} = (1-d) \tau_\varepsilon^+$$

$$\tau_\varepsilon^+ = (\bar{\sigma}^+ : \underbrace{\mathbb{C}^{-1} : \bar{\sigma}}_\varepsilon)^{\frac{1}{2}} = \sqrt{\bar{\sigma}^+ : \varepsilon}$$

Characterization of the elastic domain

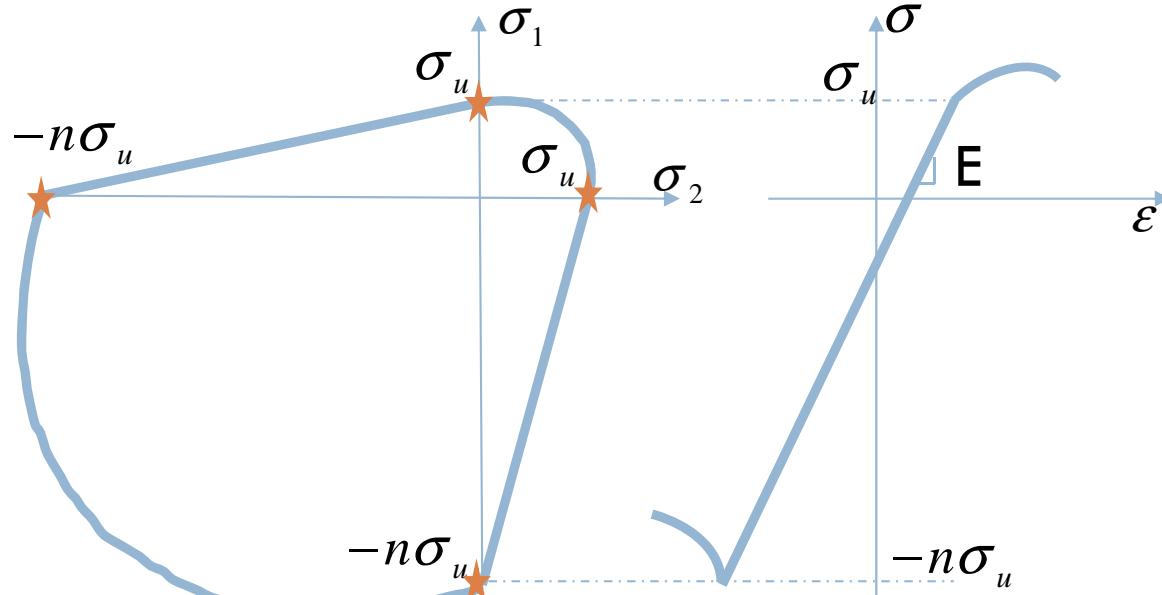
- a) If $\left. \begin{array}{l} \sigma_1 > 0 \rightarrow \sigma_1 = \sigma_1^+ \\ \sigma_2 > 0 \rightarrow \sigma_2 = \sigma_2^+ \\ \sigma_3 > 0 \rightarrow \sigma_3 = \sigma_3^+ \end{array} \right\} \Rightarrow \boxed{\begin{array}{l} \text{Pure tensile state} \rightarrow \sigma = \sigma^+ \\ \rightarrow \tau_\sigma^+ \equiv \sqrt{\sigma^+ : \mathbb{C}^{-1} : \sigma} = \tau_\sigma \end{array}}$
- b) If $\left. \begin{array}{l} \sigma_1 < 0 \rightarrow \sigma_1^+ = 0 \\ \sigma_2 < 0 \rightarrow \sigma_2^+ = 0 \\ \sigma_3 < 0 \rightarrow \sigma_3^+ = 0 \end{array} \right\} \rightarrow \boxed{\begin{array}{l} \text{Pure compressive state} \\ \sigma^+ = 0 \rightarrow \tau_\sigma^+ \equiv \sqrt{\sigma^+ : \mathbb{C}^{-1} : \sigma} = 0 \rightarrow f = \underbrace{\tau_\sigma^+}_{=0} - \sigma_u < 0 \\ \rightarrow \text{The state is always elastic} \end{array}}$



Characterization of the elastic domain

□ Non-symmetric tension-compression model

$$\left\{ \begin{array}{l} \tau_\sigma = \left[\theta + \frac{1-\theta}{n} \right] \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} \\ \theta = \frac{\sum_1^3 \langle \sigma_i \rangle}{\sum_1^3 |\sigma_i|} = \frac{\sum_1^3 \langle \bar{\sigma}_i \rangle}{\sum_1^3 |\bar{\sigma}_i|} \end{array} \right. \rightarrow \left\{ \begin{array}{l} \sigma_1, \sigma_2, \sigma_3 > 0 \rightarrow \theta = 1 \rightarrow \tau_\sigma = \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} \\ \sigma_1, \sigma_2, \sigma_3 < 0 \rightarrow \theta = 0 \rightarrow \tau_\sigma = \frac{1}{n} \sqrt{\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma}} \end{array} \right.$$



2.2 Viscodamage model

Ch.2. Continuum Damage Models

Model characterization

- The rate effects can be accommodated into the inviscid damage model (rate independent)

Evolution equation

$$\dot{r} = \lambda(\boldsymbol{\varepsilon}, r) \geq 0, \quad r|_{t=0} = r_0, \quad r \in [r_0, \infty]$$

Constitutive equation

$$\boldsymbol{\sigma} = (1-d)\mathbb{C} : \boldsymbol{\varepsilon}$$

Damage function

$$g(\boldsymbol{\varepsilon}, r) \equiv \tau_\varepsilon - r$$

Karush-Kuhn-Tucker and persistency conditions

$$\dot{r} = \lambda \geq 0, \quad g \leq 0 \quad \Rightarrow \quad \lambda g \leq 0$$

$$\text{if } g = 0 \quad \Rightarrow \quad \lambda \dot{g} \leq 0$$

Model characterization

□ Visco regularization (Perzyna's Regularization)

Obtained by replacing $\lambda \rightarrow \frac{1}{\eta} \langle g \rangle$ in the inviscid model

Evolution equation

$$\dot{r} = \lambda(\boldsymbol{\varepsilon}, r) = \frac{1}{\eta} \langle g \rangle$$

Constitutive equation

$$\boldsymbol{\sigma} = (1-d)\mathbb{C} : \boldsymbol{\varepsilon}$$

where $\eta \geq 0$ is the viscosity
The only change is the evolution law
 $\langle \rangle$ is the McAuley bracket

Damage function

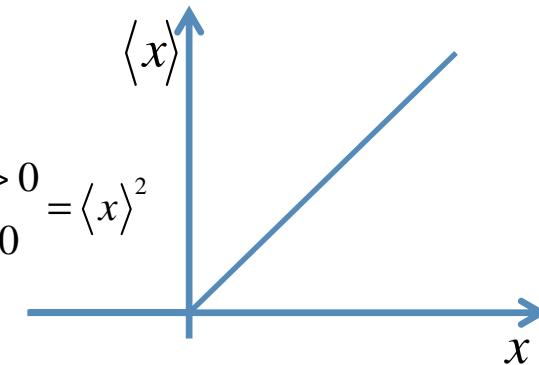
$$g(\boldsymbol{\varepsilon}, r) \equiv \tau_\varepsilon - r$$

Karush – Kuhn –Tucker and persistency conditions

Not necessary

$$\langle x \rangle = \begin{cases} x & x > 0 \\ 0 & x < 0 \end{cases}$$

$$x \langle x \rangle = \begin{cases} x^2 & x > 0 \\ 0 & x < 0 \end{cases} = \langle x \rangle^2$$



Model characterization

□ Remark 1

$$\frac{1}{\eta} \geq 0; \quad \langle g \rangle \geq 0 \quad \rightarrow$$

$$\lambda = \frac{1}{\eta} \langle g \rangle \geq 0$$

$$\lambda g = \frac{1}{\eta} \underbrace{\langle g \rangle}_{{\langle g \rangle}^2} g = \frac{1}{\eta} \underbrace{\langle g \rangle^2}_{\lambda \eta} = \frac{1}{\eta} (\lambda \eta)^2 = \eta \underbrace{\lambda^2}_{=r} = \eta r^2 \quad \text{As } \eta \rightarrow 0 \quad \lambda g = 0$$

□ Remark 2

$$\langle g \rangle = \lambda \eta \geq 0$$

$$\text{As } \eta \rightarrow 0 \quad \langle g \rangle = 0 \quad \rightarrow \quad g \leq 0$$

□ Remark 3

$$\lambda g = 0 \quad \dot{\lambda} g + \lambda \dot{g} = 0$$

$$\text{If } g = 0 \quad \rightarrow \quad \lambda \dot{g} = 0$$

$$\eta \rightarrow 0 \quad \begin{cases} \lambda \geq 0 \quad g \leq 0 \quad \lambda g = 0 \rightarrow \text{Loading/unloading conditions} \\ \text{If } g = 0 \quad \lambda \dot{g} = 0 \rightarrow \text{Persistency conditions} \end{cases}$$

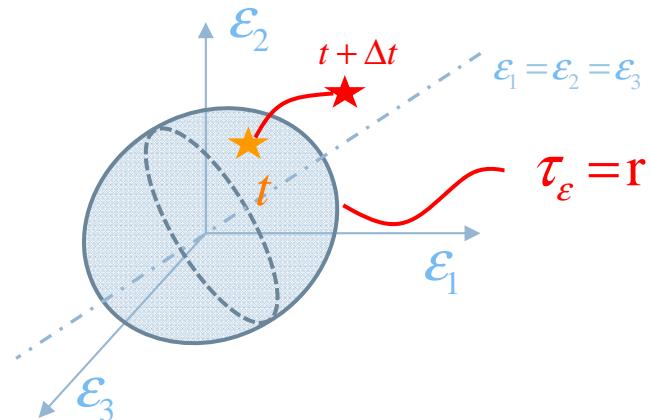
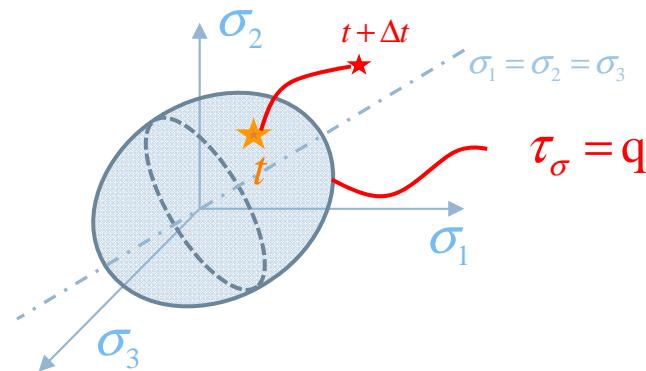
As $\eta \rightarrow 0$ (inviscid case) the rate independent continuum damage model is recovered

Model characterization

□ Remark 3

For $\eta \neq 0$ and $\lambda \neq 0$ then $\langle g \rangle = \lambda \eta \neq 0 \rightarrow \begin{cases} g \equiv \tau_\varepsilon - r > 0 \rightarrow \tau_\varepsilon > r \\ f \equiv \tau_\sigma - q > 0 \rightarrow \tau_\sigma > q \end{cases}$

The stress/strain state can lay outside the elastic domain



□ Remark 4

$$\begin{cases} f \leq 0 \Leftrightarrow g \leq 0 \rightarrow \dot{r} = \frac{1}{\eta} \langle g \rangle = 0 \rightarrow \text{No evolution (unloading / neutral loading)} \\ f > 0 \Leftrightarrow g > 0 \rightarrow \dot{r} = \frac{1}{\eta} \langle g \rangle > 0 \rightarrow \text{Evolution (loading)} \end{cases}$$

Free energy and dissipation

□ Free Energy

$$\psi(\boldsymbol{\varepsilon}, r) = (1 - d)\psi_0(\boldsymbol{\varepsilon}) = (1 - d)\frac{1}{2}(\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon})$$

□ Dissipation

$$\mathcal{D} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\psi} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}} \quad \mathcal{D} = \left(\boldsymbol{\sigma} - \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial d} : \dot{d} \geq 0 \quad \forall \dot{\boldsymbol{\varepsilon}}$$

Coleman's Theorem

$$\boldsymbol{\sigma} = \frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \frac{\partial \psi_0(\boldsymbol{\varepsilon}, d(r))}{\partial \boldsymbol{\varepsilon}} = (1 - d(r)) \mathbb{C} : \boldsymbol{\varepsilon}$$

$$\mathcal{D} = -\frac{\partial \psi(\boldsymbol{\varepsilon}, d(r))}{\partial d} : \dot{d} = \underbrace{\psi_0}_{\geq 0} : \dot{d} \geq 0 \quad \xrightarrow{\hspace{1cm}} \quad \boxed{\dot{d} \geq 0}$$

Tangent constitutive tensor

- Tangent constitutive operator

$$\sigma = \sigma(\varepsilon, t) \quad \left\{ \begin{array}{ll} \sigma = \sigma(\varepsilon(t)) & \text{Inviscid model (t as a parameter)} \\ \sigma = \sigma(\varepsilon(t), t) & \text{Viscous model (t as an independent variable)} \end{array} \right.$$

$$\dot{\sigma} = (1-d)\mathbb{C} : \dot{\varepsilon} - \dot{d} \underbrace{\mathbb{C} : \varepsilon}_{\bar{\sigma}} = (1-d)\mathbb{C} : \dot{\varepsilon} - \dot{d}\bar{\sigma}$$

$$\dot{\sigma} = (1 - d(r))\mathbb{C} : \dot{\varepsilon} - \underbrace{d'(r)\dot{r}}_{\dot{d}} \bar{\sigma} = (1 - d(r))\mathbb{C} : \dot{\varepsilon} - d'(r) \underbrace{\dot{r}}_{\frac{1}{\eta} \langle g \rangle} \bar{\sigma}$$

$$\dot{\sigma} = (1 - d(r))\mathbb{C} : \dot{\varepsilon} - \frac{1}{\eta} d'(r) \langle g(\varepsilon, r) \rangle \bar{\sigma}$$

$$\dot{\sigma} = \frac{d\sigma(\varepsilon, t)}{dt} = \frac{\partial \sigma}{\partial \varepsilon} : \dot{\varepsilon} + \frac{\partial \sigma}{\partial t}$$

$$\mathbb{C}_{\text{tang}}^{vd} = \frac{\partial \sigma}{\partial \varepsilon} = (1 - d) \mathbb{C}$$

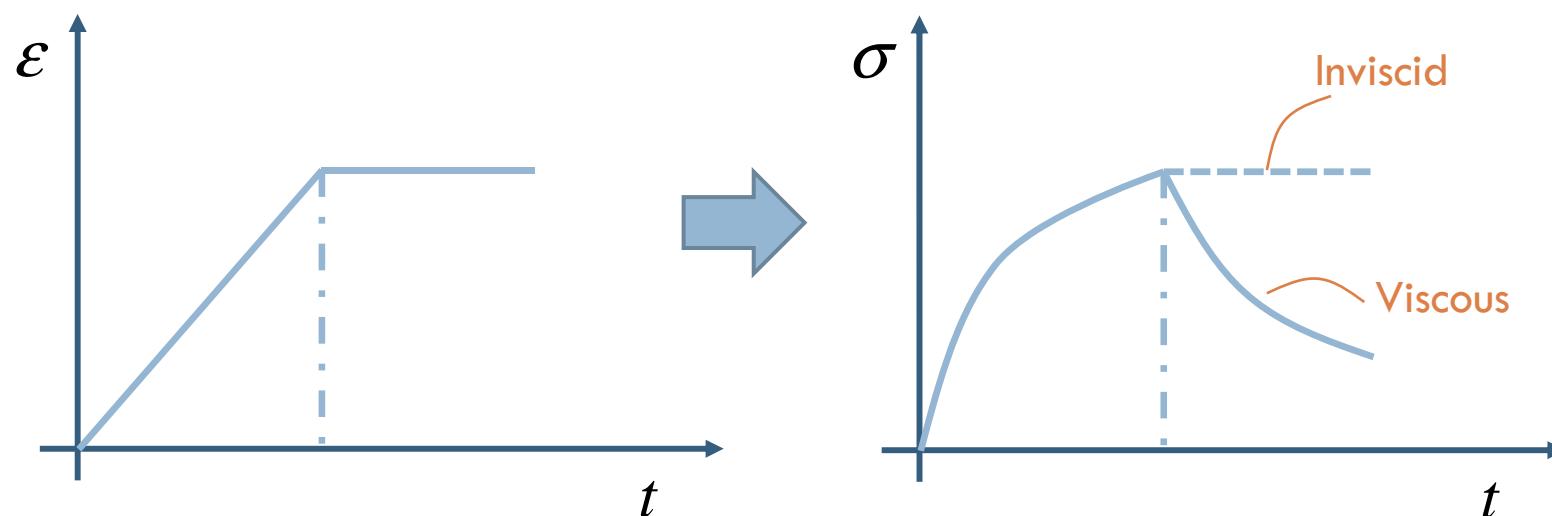
$$\frac{\partial \sigma}{\partial t} = \frac{1}{\eta} d'(r) \langle g(\varepsilon, r) \rangle \mathbb{C} : \varepsilon$$

Time dependency/Rate dependency

□ Remark

$$\sigma = \sigma(\varepsilon(t), t)$$

The stress tensor can change even if ε remains constant





END OF LECTURE 4