

# Numerical integration

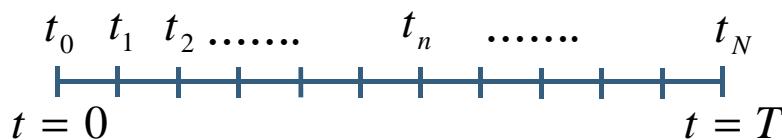
## □ Analytical integration

$$\dot{r} = \frac{1}{\eta} \langle \tau_{\varepsilon}(\boldsymbol{\varepsilon}(t)) - r(t) \rangle \rightarrow \begin{cases} \text{Data : } \boldsymbol{\varepsilon}(s) \\ \qquad \qquad s \in [0, t] \\ \text{Result: } r(t) \end{cases}$$

1<sup>st</sup> order ordinary differential equation  
Analytical solution not always available

$$\rightarrow d(r) = 1 - \frac{q(r)}{r} \rightarrow \boxed{\boldsymbol{\sigma} = (1 - d) \mathbb{C} : \boldsymbol{\varepsilon}}$$

## □ Numerical integration



$$\left\{ \begin{array}{l} [0, T] = \bigcup_{n=1}^{n=N} [t_n, t_{n+1}] \\ [t_n, t_{n+1}] \rightarrow \text{Time interval/step "n+1"} \\ \Delta t = t_{n+1} - t_n \rightarrow \text{Time increment} \end{array} \right.$$

$$x(t_{n+1}) \equiv x_{n+1} = g(x_n, x_{n-1}, x_{n-2}, \dots)$$

Approximate solution (depending on the values at previous discrete times)

# Numerical integration

## □ Alpha method

$$\dot{x} = f(x(t), t)$$

Equation to solve

Defining

$$\begin{cases} x_n := x(t_n) \\ x_{n+1} := x(t_{n+1}) \\ x_{n+\alpha} := x(t_n + \alpha\Delta t) \end{cases}$$

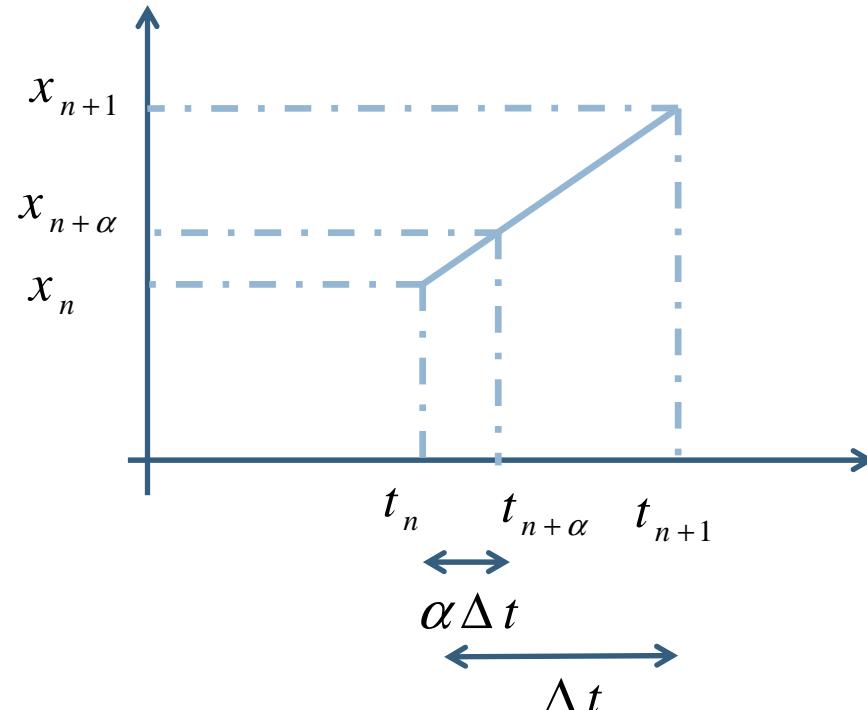
Approximation:

$$x_{n+\alpha} \simeq (1 - \alpha)x_n + \alpha x_{n+1}$$

## □ Integration

$$\dot{x}_{n+1} = \frac{x_{n+1} - x_n}{\Delta t} = f(x_{n+\alpha}, t_{n+\alpha})$$

$$\begin{cases} \alpha = 0 & x_{n+\alpha} = x_n \text{ Forward Euler (explicit) scheme} \\ \alpha = 0.5 & x_{n+\alpha} = \frac{x_{n+1} + x_n}{2} \text{ Crank-Nicholson (mid point rule) scheme} \\ \alpha = 1 & x_{n+\alpha} = x_{n+1} \text{ Backward Euler (implicit) scheme} \end{cases}$$



# Numerical integration

$$\dot{r} = \frac{1}{\eta} \langle \tau_{\varepsilon}( \boldsymbol{\varepsilon}(t) ) - r(t) \rangle$$

□ Data  $\begin{cases} r|_{t=t_0} = r_0 \\ r_n, \boldsymbol{\varepsilon}_n, \boldsymbol{\varepsilon}_{n+1} \Rightarrow \tau_{\varepsilon_n}(\boldsymbol{\varepsilon}_n), \tau_{\varepsilon_{n+1}}(\boldsymbol{\varepsilon}_{n+1}) \end{cases}$

□ Unknowns

$$r_{n+1} \rightarrow q_{n+1} \rightarrow d_{n+1} \rightarrow \sigma_{n+1}$$

□ Integration  $\begin{cases} \dot{r}_{n+1} = \frac{r_{n+1} - r_n}{\Delta t} = \frac{1}{\eta} \langle \tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha} \rangle \\ \tau_{\varepsilon_{n+\alpha}} = (1-\alpha)\tau_{\varepsilon_n} + \alpha\tau_{\varepsilon_{n+1}} \rightarrow Data \\ r_{n+\alpha} = (1-\alpha)r_n + \alpha r_{n+1} \end{cases}$

# Integration of the constitutive equation: Identification of the current state

**THEOREM:**  $\boxed{\tau_{\varepsilon_{n+a}} \leq r_{n+\alpha} \Leftrightarrow \tau_{\varepsilon_{n+a}} \leq r_n} \rightarrow (\text{equivalence})$

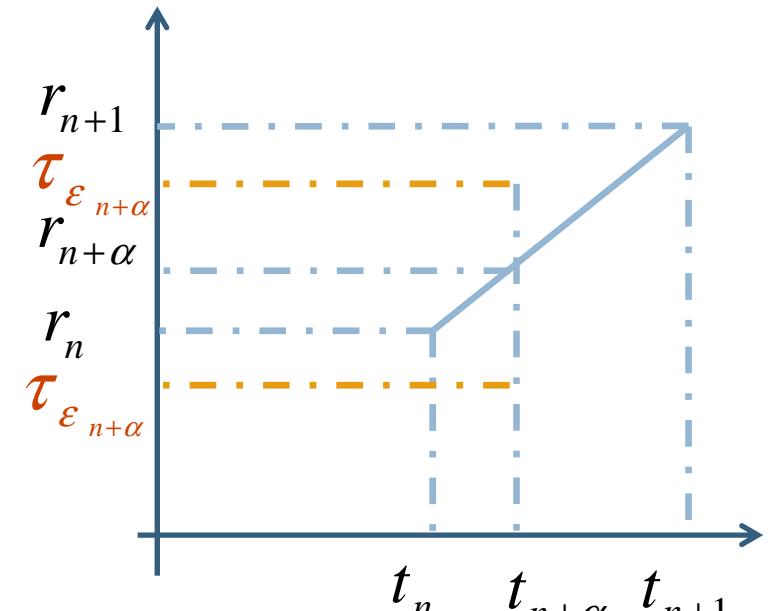
Proof: a)  $\tau_{\varepsilon_{n+a}} \leq r_{n+\alpha} \Rightarrow \tau_{\varepsilon_{n+a}} \leq r_n$

$$\begin{cases} \dot{r}_{n+1} = \frac{r_{n+1} - r_n}{\Delta t} = \frac{1}{\eta} \underbrace{\langle \tau_{\varepsilon_{n+a}} - r_{n+\alpha} \rangle}_{=0} = 0 \Rightarrow r_{n+1} = r_n \\ r_{n+1} = r_n \Rightarrow r_{n+\alpha} = (1-\alpha)r_n + \alpha \underbrace{r_n}_{r_{n+\alpha}} = r_n \end{cases}$$

$$\tau_{\varepsilon_{n+a}} \leq r_{n+\alpha} \Rightarrow \boxed{\tau_{\varepsilon_{n+a}} \leq r_n}$$

b)  $\tau_{\varepsilon_{n+a}} \leq r_n \Rightarrow \tau_{\varepsilon_{n+a}} \leq r_{n+\alpha}$

$$\tau_{\varepsilon_{n+a}} \leq r_n \leq r_{n+\alpha} \Rightarrow \boxed{\tau_{\varepsilon_{n+a}} \leq r_{n+\alpha}}$$



**COROLLARY:**  $\boxed{\tau_{\varepsilon_{n+a}} > r_{n+\alpha} \Leftrightarrow \tau_{\varepsilon_{n+a}} > r_n} \rightarrow (\text{equivalence})$

Proof: a)  $\tau_{\varepsilon_{n+a}} > r_{n+\alpha} \Rightarrow \tau_{\varepsilon_{n+a}} \not\leq r_{n+\alpha} \Rightarrow \tau_{\varepsilon_{n+a}} \not\leq r_n \Rightarrow \tau_{\varepsilon_{n+a}} > r_n$

b)  $\tau_{\varepsilon_{n+a}} > r_n \Rightarrow \tau_{\varepsilon_{n+a}} \not\leq r_n \Rightarrow \tau_{\varepsilon_{n+a}} \not\leq r_{n+\alpha} \Rightarrow \tau_{\varepsilon_{n+a}} > r_{n+\alpha}$

# Integration of the constitutive equation: Identification of the current state

□ State identified at time  $t_{n+\alpha}$  in the current time interval  $[t_n, t_{n+1}]$

## 1 Unloading / Neutral loading

$$g_{n+\alpha} = \tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha} \leq 0 \Leftrightarrow \tau_{\varepsilon_{n+\alpha}} \leq r_{n+\alpha} \stackrel{\text{Theorem}}{\Leftrightarrow} \boxed{\tau_{\varepsilon_{n+\alpha}} \leq r_n}$$

$$\dot{r}_{n+1} = \frac{r_{n+1} - r_n}{\Delta t} = \frac{1}{\eta} \underbrace{\langle \tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha} \rangle}_{=0} = 0 \rightarrow \boxed{r_{n+1} = r_n}$$

## 2 Loading

$$g_{n+\alpha} = \tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha} > 0 \Leftrightarrow \tau_{\varepsilon_{n+\alpha}} > r_{n+\alpha} \stackrel{\text{Corollary}}{\Leftrightarrow} \boxed{\tau_{\varepsilon_{n+\alpha}} > r_n}$$

$$\dot{r}_{n+1} = \frac{r_{n+1} - r_n}{\Delta t} = \frac{1}{\eta} \langle \tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha} \rangle = \frac{1}{\eta} (\tau_{\varepsilon_{n+\alpha}} - r_{n+\alpha}) > 0$$

$$r_{n+1} = r_n + \frac{\Delta t}{\eta} (\tau_{\varepsilon_{n+\alpha}} - [(1-\alpha)r_n + \alpha r_{n+1}]) \rightarrow \boxed{r_{n+1} = \frac{[\eta - \Delta t(1-\alpha)]r_n + \Delta t\tau_{\varepsilon_{n+\alpha}}}{\eta + \alpha\Delta t}}$$

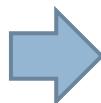
# Inviscid problem

## ■ Inviscid case and implicit integration

$\eta \rightarrow 0$   
Inviscid  
case

$\alpha = 1$   
Implicit  
integration

$$r_{n+1} = \frac{[\eta - \Delta t(1 - \alpha)] r_n + \Delta t \tau_{\varepsilon_{n+\alpha}}}{\eta + \alpha \Delta t}$$



$$r_{n+1} = \tau_{\varepsilon_{n+1}}$$

Numerical integration inherits the properties of the model and recovers the solution of the rate independent problem for the **inviscid case and implicit integration**

# Numerical integration: stability analysis

$$\begin{cases} f(x(t), \dot{x}(t)) = 0 \\ x|_{t=0} = x_0 \end{cases} \rightarrow x_{n+1} = ax_n + b$$

$|a| \leq 1$  → { errors do not propagate  
(stable integration)

$$r_{n+1} = \frac{[\eta - \Delta t(1 - \alpha)]r_n + \Delta t\tau_{\varepsilon_{n+1}}}{\eta + \alpha\Delta t} = \underbrace{\frac{[\eta - \Delta t(1 - \alpha)]}{\eta + \alpha\Delta t} r_n}_a + \underbrace{\frac{\Delta t\tau_{\varepsilon_{n+1}}}{\eta + \alpha\Delta t}}_b$$

$$|a| = \left| \frac{[\eta - \Delta t(1 - \alpha)]}{\eta + \alpha\Delta t} \right| \leq 1 \quad \rightarrow \quad -1 \leq \frac{[\eta - \Delta t(1 - \alpha)]}{\eta + \alpha\Delta t} \leq 1$$

$$\frac{[\eta - \Delta t(1 - \alpha)]}{\eta + \alpha\Delta t} \leq 1 \quad \rightarrow \quad \eta \geq 0 \quad \text{Cranck-Nicholson (mid-point rule)}$$

$$\frac{[\eta - \Delta t(1 - \alpha)]}{\eta + \alpha\Delta t} \geq -1 \quad \rightarrow \quad \alpha \geq \frac{1}{2} \quad \rightarrow \quad \boxed{\frac{1}{2} \leq \alpha \leq 1}$$

Backward-Euler

# Numerical integration: accuracy analysis

## □ Accuracy

$$\dot{r}(t) = \frac{1}{\eta} (\tau_\varepsilon(t) - r(t))$$

$$\ddot{r}(t) = \frac{1}{\eta} (\dot{\tau}_\varepsilon(t) - \dot{r}(t))$$

□ .....

## □ Alpha method

$$r_{n+\alpha} = (1 - \alpha)r_n + \alpha r_{n+1} \quad \rightarrow \quad r_{n+\alpha}(t_{n+1}) = (1 - \alpha)r_n + \alpha r_{n+1}(t_{n+1})$$

$$\dot{r}_{n+\alpha} = \frac{\partial r_{n+\alpha}}{\partial t_{n+1}} = \alpha \frac{\partial r_{n+1}}{\partial t_{n+1}} = \alpha \dot{r}_{n+1} ; \quad \ddot{r}_{n+\alpha} = \frac{\partial \dot{r}_{n+\alpha}}{\partial t_{n+1}} = \alpha \frac{\partial \dot{r}_{n+1}}{\partial t_{n+1}} = \alpha \ddot{r}_{n+1}$$

$$\tau_{n+\alpha} = (1 - \alpha)\tau_n + \alpha \tau_{n+1} \quad \rightarrow$$

$$\dot{\tau}_{n+\alpha} = \frac{\partial \tau_{n+\alpha}}{\partial t_{n+1}} = \alpha \dot{\tau}_{n+1} ; \quad \ddot{\tau}_{n+\alpha} = \frac{\partial \dot{\tau}_{n+\alpha}}{\partial t_{n+1}} = \alpha \frac{\partial \dot{\tau}_{n+1}}{\partial t_{n+1}} = \alpha \ddot{\tau}_{n+1}$$

# Numerical integration: accuracy analysis

## □ Alpha method

$$\dot{r}_{n+1} = \frac{r_{n+1} - r_n}{\Delta t} = \frac{1}{\eta} (\tau_{n+\alpha} - r_{n+\alpha}) \quad \rightarrow$$

$$r_{n+1}(t_{n+1}) = r_n + \frac{\Delta t}{\eta} (\tau_{n+\alpha} - r_{n+\alpha}) = r_n + \frac{t_{n+1} - t_n}{\eta} [\tau_{n+\alpha}(t_{n+1}) - r_{n+\alpha}(t_{n+1})]$$

Taking time derivatives:

$$\dot{r}_{n+1} = \frac{1}{\eta} (\tau_{n+\alpha} - r_{n+\alpha}) + \overbrace{\frac{t_{n+1} - t_n}{\eta}}^{\Delta t} \left[ \underbrace{\dot{\tau}_{n+\alpha}}_{\alpha \dot{\tau}_{n+1}} - \underbrace{\dot{r}_{n+\alpha}}_{\alpha \dot{r}_{n+1}} \right]$$

$$\ddot{r}_{n+1} = \frac{1}{\eta} \left[ \underbrace{\dot{\tau}_{n+\alpha}}_{\alpha \dot{\tau}_{n+1}} - \underbrace{\dot{r}_{n+\alpha}}_{\alpha \dot{r}_{n+1}} \right] + \frac{1}{\eta} \left[ \underbrace{\dot{\tau}_{n+\alpha}}_{\alpha \dot{\tau}_{n+1}} - \underbrace{\dot{r}_{n+\alpha}}_{\alpha \dot{r}_{n+1}} \right] + \overbrace{\frac{t_{n+1} - t_n}{\eta}}^{\Delta t} \left[ \underbrace{\dot{\tau}_{n+\alpha}}_{\alpha \dot{\tau}_{n+1}} - \underbrace{\ddot{r}_{n+\alpha}}_{\alpha \ddot{r}_{n+1}} \right]$$

# Numerical integration: accuracy analysis

$$\dot{r}(t) = \frac{1}{\eta}(\tau_\varepsilon(t) - r(t)) ; \quad \ddot{r}(t) = \frac{1}{\eta}(\dot{\tau}_\varepsilon(t) - \dot{r}(t))$$

Taking the limit  $\Delta t \rightarrow 0$  ( $t_{n+1} = t_{n+\alpha} \rightarrow t_n$ )

$$\dot{r}_{n+1} = \frac{1}{\eta}(\tau_{n+\alpha} - r_{n+\alpha}) + \frac{\alpha \Delta t}{\eta}(\dot{\tau}_{n+1} - \dot{r}_{n+1}) \rightarrow \dot{r}_{n+1} = \frac{1}{\eta}(\tau_{n+1} - r_{n+1}) \rightarrow \text{holds}$$

$$\left\{ \begin{array}{l} \ddot{r}_{n+1} = \frac{2\alpha}{\eta}(\dot{\tau}_{n+\alpha} - \dot{r}_{n+\alpha}) + \frac{\Delta t}{\eta}(\ddot{\tau}_{n+1} - \ddot{r}_{n+1}) \rightarrow \\ \rightarrow \ddot{r}_{n+1} = \frac{2\alpha}{\eta}(\dot{\tau}_{n+1} - \dot{r}_{n+1}) \end{array} \right. \rightarrow \text{holds only for } \alpha = \frac{1}{2}$$

## □ Alpha method: Summary of stability/integration analysis results

Stable:  $\alpha = [\frac{1}{2}, 1]$  First order accurate:  $\alpha = [0, 1]$  Second order accurate:  $\alpha = \frac{1}{2}$

# Consistent (algorithmic) tangent operator

$$\begin{cases} \mathbb{C}_{\text{tang}}^{vd} = \frac{\partial \boldsymbol{\sigma}(\boldsymbol{\varepsilon})}{\partial \boldsymbol{\varepsilon}} = (1-d)\mathbb{C} \rightarrow \text{Analytic tangent operator} \\ \mathbb{C}_{\text{alg},n+1}^{vd} := \frac{\partial \boldsymbol{\sigma}_{n+1}(\boldsymbol{\varepsilon}_{n+1})}{\partial \boldsymbol{\varepsilon}_{n+1}} \rightarrow \text{Algorithmic tangent operator} \end{cases}$$

$$\begin{cases} \mathbb{C}_{\text{alg},n+1}^{vd} \neq \mathbb{C}_{\text{tang}}^{vd}(\boldsymbol{\varepsilon}_{n+1}) \\ \lim_{\Delta t \rightarrow 0} \mathbb{C}_{\text{alg},n+1}^{vd} = \mathbb{C}_{\text{tang}}^{vd}(\boldsymbol{\varepsilon}_{n+1}) \end{cases}$$

$$\begin{cases} \boldsymbol{\sigma}_{n+1} = [1 - d_{n+1}] \mathbb{C} : \boldsymbol{\varepsilon}_{n+1} \\ d_{n+1}(r_{n+1}) = 1 - \frac{q(r_{n+1})}{r_{n+1}} \\ r_{n+1} = \begin{cases} r_n & (\text{elastic/unloading}) \\ \frac{[\eta - \Delta t(1-\alpha)]}{\eta + \alpha\Delta t} r_n + \frac{\Delta t}{\eta + \alpha\Delta t} \tau_{\varepsilon_{n+\alpha}}(\boldsymbol{\varepsilon}_{n+1}) & (\text{loading}) \end{cases} \end{cases}$$

$$\rightarrow \begin{cases} \boldsymbol{\sigma}_{n+1} = \boldsymbol{\Sigma}_{n+1}(\boldsymbol{\varepsilon}_{n+1}) \\ \delta\boldsymbol{\sigma}_{n+1} = \mathbb{C}_{\text{alg},n+1}^{vd}(\boldsymbol{\varepsilon}_{n+1}) : \delta\boldsymbol{\varepsilon}_{n+1} \end{cases}$$

# Consistent (algorithmic) tangent operator

## □ Stress differentiation

$$\delta\boldsymbol{\sigma}_{n+1} = (1-d_{n+1})\mathbb{C} : \delta\boldsymbol{\epsilon}_{n+1} - d'(r_{n+1})\delta r_{n+1} \otimes \underbrace{\mathbb{C} : \boldsymbol{\epsilon}_{n+1}}_{\bar{\boldsymbol{\sigma}}_{n+1}} ; \quad \delta r_{n+1} = \begin{cases} 0 & \text{(elastic/unloading)} \\ \frac{\alpha\Delta t}{\eta + \alpha\Delta t} \delta\tau_{\boldsymbol{\epsilon}_{n+1}} & \text{(loading)} \end{cases}$$

$$\tau_{\boldsymbol{\epsilon}_{n+1}} = \sqrt{\boldsymbol{\epsilon}_{n+1} : \mathbb{C} : \boldsymbol{\epsilon}_{n+1}} \quad \Rightarrow \quad \delta\tau_{\boldsymbol{\epsilon}_{n+1}} = \frac{1}{\tau_{\boldsymbol{\epsilon}_{n+1}}} \underbrace{\boldsymbol{\epsilon}_{n+1} : \mathbb{C} : \delta\boldsymbol{\epsilon}_{n+1}}_{\bar{\boldsymbol{\sigma}}_{n+1}} = \frac{1}{\tau_{\boldsymbol{\epsilon}_{n+1}}} \bar{\boldsymbol{\sigma}}_{n+1} : \delta\boldsymbol{\epsilon}_{n+1}$$

$$\delta\boldsymbol{\sigma}_{n+1} = \begin{cases} (1-d_n)\mathbb{C} : \delta\boldsymbol{\epsilon}_{n+1} & \text{(elastic/unloading)} \\ [(1-d_{n+1})\mathbb{C} - \frac{\alpha\Delta t}{\eta + \alpha\Delta t} \frac{1}{\tau_{\boldsymbol{\epsilon}_{n+1}}} d'_{n+1} \bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}] : \delta\boldsymbol{\epsilon}_{n+1} & \text{(loading)} \end{cases} = \mathbb{C}_{\text{alg},n+1}^{vd} : \delta\boldsymbol{\epsilon}_{n+1}$$

$$\mathbb{C}_{\text{alg},n+1}^{vd} = \underbrace{(1-d_n)\mathbb{C}}_{\mathbb{C}_{\text{tang},n+1}^{vd}} \quad \underbrace{[(1-d_{n+1})\mathbb{C} - \frac{\alpha\Delta t}{\eta + \alpha\Delta t} \frac{1}{\tau_{\boldsymbol{\epsilon}_{n+1}}} d'_{n+1} \bar{\boldsymbol{\sigma}}_{n+1} \otimes \bar{\boldsymbol{\sigma}}_{n+1}]}_{\text{Additional term} \rightarrow \mathcal{O}(\Delta t)} \quad (d'_{n+1} = -\frac{H_{n+1}r_{n+1} - q(r_{n+1})}{(r_{n+1})^2})$$

# Consistent (algorithmic) tangent operator

$$\mathbb{C}_{\text{alg},n+1}^{\text{vd}} = \begin{cases} (1-d_n)\mathbb{C} & (\text{elastic/unloading}) \\ \underbrace{(1-d_{n+1})\mathbb{C}}_{\mathbb{C}_{\text{tang},n+1}^{\text{vd}}} - \underbrace{\frac{\alpha\Delta t}{\eta + \alpha\Delta t} \frac{1}{\tau_{\varepsilon_{n+1}}} d'_{n+1} \bar{\sigma}_{n+1} \otimes \bar{\sigma}_{n+1}}_{\text{Additional term } \rightarrow \mathcal{O}(\Delta t)} & (\text{loading}) \end{cases}$$

## □ Remark 1

If:  $\alpha = 0$

$$\mathbb{C}_{\text{alg},n+1}^{\text{vd}} = \mathbb{C}_{\text{tang},n+1}^{\text{vd}}$$

If:  $\Delta t = 0$

$$\mathbb{C}_{\text{alg},n+1}^{\text{vd}} = \mathbb{C}_{n+1}^d$$

Algorithmic and analytical tangent operators match

## □ Remark 2

If:  $\eta \rightarrow 0$

$$\mathbb{C}_{\text{alg},n+1}^{\text{vd}} = (1-d_{n+1})\mathbb{C} - \frac{1}{\tau_{\varepsilon_{n+1}}} d'_{n+1} \bar{\sigma}_{n+1} \otimes \bar{\sigma}_{n+1}$$

Algorithmic viscous tangent operator match (in the inviscid limit) the one for the rate independent damage model

# Integration of the constitutive equation

## Numerical algorithm:

INPUT DATA  $[t_n, t_n + \Delta t = t_{n+1}] \rightarrow \boldsymbol{\varepsilon}_n, r_n, \boldsymbol{\varepsilon}_{t_{n+1}}$

Step 1 → Compute  $\left\{ \begin{array}{l} \bar{\boldsymbol{\sigma}}_{n+1} = \mathbb{C} : \boldsymbol{\varepsilon}_{n+1} \rightarrow \left\{ \begin{array}{l} \tau_{\boldsymbol{\varepsilon}_n} = \sqrt{\boldsymbol{\varepsilon}_n : \mathbb{C} : \boldsymbol{\varepsilon}_n} \\ \tau_{\boldsymbol{\varepsilon}_{n+1}} = \sqrt{\boldsymbol{\varepsilon}_{n+1} : \mathbb{C} : \boldsymbol{\varepsilon}_{n+1}} = \sqrt{\bar{\boldsymbol{\sigma}}_{n+1} : \boldsymbol{\varepsilon}_{n+1}} \end{array} \right. \\ \tau_{\boldsymbol{\varepsilon}_{n+\alpha}} = (1 - \alpha)\tau_{\boldsymbol{\varepsilon}} + \alpha\tau_{\boldsymbol{\varepsilon}_{n+1}} \end{array} \right.$



Step 2 → If  $\tau_{\boldsymbol{\varepsilon}_{n+\alpha}} \leq r_n \rightarrow$  Elastic  
Unloading

$\rightarrow \left\{ \begin{array}{l} r_{n+1} = r_n ; d_{n+1} = d_n = 1 - \frac{q(r_{n+1})}{r_{n+1}} ; \boldsymbol{\sigma}_{n+1} = (1 - d_{n+1})\bar{\boldsymbol{\sigma}}_{n+1} \\ \mathbb{C}_{\text{alg}, n+1}^{vd} = (1 - d_{n+1})\mathbb{C} \end{array} \right. \rightarrow \text{EXIT}$



# Integration of the constitutive equation

## Numerical algorithm:



Step 3 → If  $\tau_{\varepsilon_{n+\alpha}} > r_n \rightarrow$  (Loading)

$$\begin{aligned} r_{n+1} &= \frac{[\eta - \Delta t(1 - \alpha)]}{\eta + \alpha \Delta t} r_n + \frac{\Delta t}{\eta + \alpha \Delta t} \tau_{\varepsilon_{n+\alpha}} ; \quad d_{n+1} = 1 - \frac{q(r_{n+1})}{r_{n+1}} \\ \rightarrow \quad \sigma_{n+1} &= (1 - d_{n+1}) \bar{\sigma}_{n+1} \\ \mathbb{C}_{\text{alg}, n+1}^{vd} &= (1 - d_{n+1}) \mathbb{C} + \\ &+ \frac{\alpha \Delta t}{\eta + \alpha \Delta t} \frac{1}{\tau_{\varepsilon_{n+1}}} \frac{H_{n+1} r_{n+1} - q(r_{n+1})}{(r_{n+1})^2} (\bar{\sigma}_{n+1} \otimes \bar{\sigma}_{n+1}) \end{aligned}$$

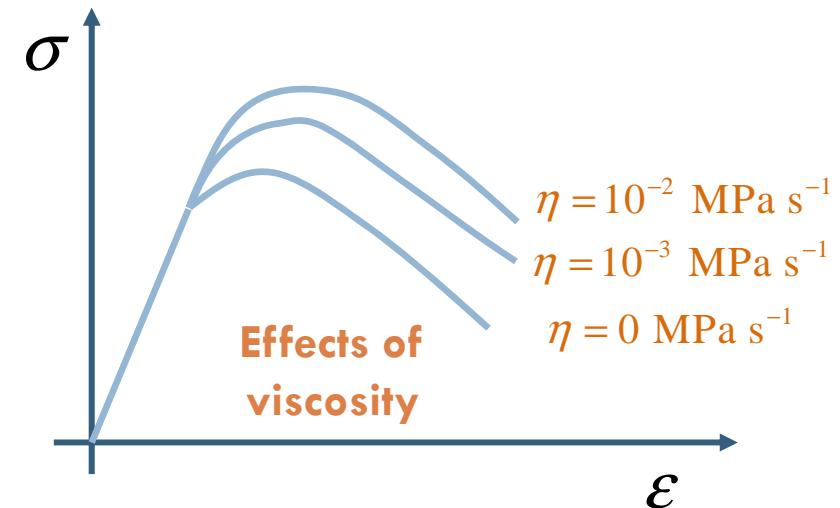
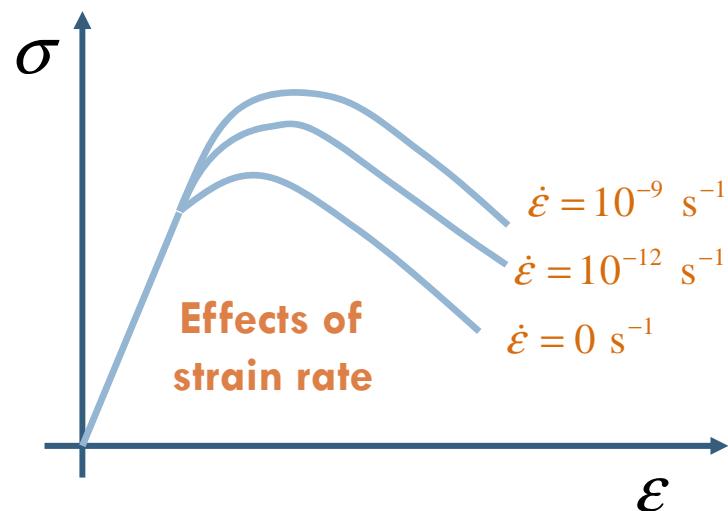
→ EXI

OUTPUT DATA  $[t_n, t_n + \Delta t = t_{n+1}] \rightarrow r_{n+1}, \sigma_{n+1}, \mathbb{C}_{\text{alg}, n+1}^{vd}$

For  $\eta = 0$  and  $\alpha = 1$  the inviscid model is recovered !!!!

# Strain-rate – Viscosity dependencies

- The strain rate and the viscosity have similar effects on the strain-stress relation



# Formulation in Voigt's Notation

- Taking into account the symmetry of the stress and strain tensors, they can be written in vector form:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_z \end{bmatrix} \stackrel{\text{not.}}{=} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} \xrightarrow{\hspace{1cm}} \{\boldsymbol{\varepsilon}\}^{\text{def}} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \in \mathbb{R}^6$$

## REMARK

The double contraction  $(\boldsymbol{\sigma} : \boldsymbol{\varepsilon})$  is equivalent to the scalar (dot) product  $(\{\boldsymbol{\sigma}\} \cdot \{\boldsymbol{\varepsilon}\})$ :

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \{\boldsymbol{\sigma}\} \cdot \{\boldsymbol{\varepsilon}\} \quad \text{vectors} \quad \Leftrightarrow \quad \sigma_{ij} \varepsilon_{ij} = \sigma_i \varepsilon_i$$

2<sup>nd</sup> order tensors

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \xrightarrow{\hspace{1cm}} \{\boldsymbol{\sigma}\}^{\text{def}} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} \in \mathbb{R}^6$$

## VOIGT'S NOTATION

# Hooke's Law

□ Hooke's law in terms of the stress and strain vectors:

$$\begin{cases} \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\epsilon} \\ \mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbb{I} \end{cases} \quad \rightarrow \quad \begin{cases} \{\boldsymbol{\sigma}\} = \mathbf{D} \cdot \{\boldsymbol{\epsilon}\} \\ \sigma_i = D_{ij} \epsilon_j \quad i \in \{1, \dots, 6\} \end{cases}$$

Where  $\mathbf{D}$  is the matrix of elastic constants:

$$\mathbf{D} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

$$\left[ \begin{array}{cccccc} 1 & \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & 1 & \frac{\nu}{1-\nu} & 0 & 0 & 0 \\ \frac{\nu}{1-\nu} & \frac{\nu}{1-\nu} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{array} \right]$$

# Inverse elastic constitutive equation

$$\begin{cases} \boldsymbol{\varepsilon} = \mathbb{C}^{-1} : \boldsymbol{\sigma} \\ \mathbb{C} = -\frac{\nu}{E} \mathbf{1} \otimes \mathbf{1} + \frac{1+\nu}{2E} \mathbb{I} \end{cases} \quad \rightarrow \quad \begin{cases} \{\boldsymbol{\varepsilon}\} = \mathbf{D}^{-1} \cdot \{\boldsymbol{\sigma}\} \\ \varepsilon_j = (\mathbf{D}^{-1})_{ij} \sigma_i \quad i \in \{1, \dots, 6\} \end{cases}$$

Where  $\mathbf{D}^{-1}$  is the elastic compliance matrix:

$$\mathbf{D}^{-1} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} ; \quad G = \frac{E}{2(1+\nu)}$$

# Algorithm computations in Voigt's notation

- The expressions used in the algorithms of the model read:

$$\tau_{\varepsilon} = \sqrt{\boldsymbol{\varepsilon} : \mathbb{C} : \boldsymbol{\varepsilon}} = \sqrt{\bar{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}} = \sqrt{\{\bar{\boldsymbol{\sigma}}\} \cdot \{\boldsymbol{\varepsilon}\}} = \sqrt{\{\boldsymbol{\varepsilon}\} \cdot \{\mathbf{D}\} \cdot \{\boldsymbol{\varepsilon}\}} = \sqrt{\{\bar{\boldsymbol{\sigma}}\} \cdot \{\mathbf{D}^{-1}\} \cdot \{\bar{\boldsymbol{\sigma}}\}}$$

$$\{\boldsymbol{\sigma}\}_{n+1} = (1 - d(r_{n+1})) \{\bar{\boldsymbol{\sigma}}\}$$

$$\{\bar{\boldsymbol{\sigma}}\}_{n+1} = \mathbf{D} \cdot \{\boldsymbol{\varepsilon}\}_{n+1}$$

$$\tau_{\varepsilon_{n+1}} = \sqrt{\{\boldsymbol{\varepsilon}\}_{n+1} \cdot \mathbf{D} \cdot \{\boldsymbol{\varepsilon}\}_{n+1}} = \sqrt{\{\bar{\boldsymbol{\sigma}}\}_{n+1} \cdot \mathbf{D}^{-1} \cdot \{\bar{\boldsymbol{\sigma}}\}_{n+1}} = \sqrt{\{\bar{\boldsymbol{\sigma}}\}_{n+1} \cdot \{\boldsymbol{\varepsilon}\}_{n+1}}$$

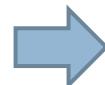
$$\underbrace{\mathbf{D}_{\text{alg},n+1}^{vd}}_{6 \times 6} = \underbrace{\frac{\partial \{\boldsymbol{\sigma}\}_{n+1}}{\partial \{\boldsymbol{\varepsilon}\}_{n+1}}}_{6 \times 6} = (1 - d_{n+1}) \underbrace{\mathbf{D}}_{6 \times 6} -$$

$$-\frac{\alpha \Delta t}{\eta + \alpha \Delta t} \frac{1}{\tau_{\varepsilon_{n+1}}} \frac{H_{n+1} r_{n+1} - q(r_{n+1})}{(r_{n+1})^2} (\underbrace{\{\bar{\boldsymbol{\sigma}}\}_{n+1}}_{6 \times 1} \otimes \underbrace{\{\bar{\boldsymbol{\sigma}}\}_{n+1}}_{1 \times 6})$$

# 2D Formulation in Voigt's Notation

- Hooke's law in terms of the stress and strain vectors:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_y \end{bmatrix} \text{not} \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y \end{bmatrix}$$



$$\{\boldsymbol{\varepsilon}\} \stackrel{\text{def}}{=} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \in \mathbb{R}^3$$

Where  $\quad$  is the matrix  
of elastic constants:

## REMARK

The double contraction  $(\boldsymbol{\sigma} : \boldsymbol{\varepsilon})$  is equivalent to the scalar (dot) product  $(\{\boldsymbol{\sigma}\} \cdot \{\boldsymbol{\varepsilon}\})$ :

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon} = \{\boldsymbol{\sigma}\} \cdot \{\boldsymbol{\varepsilon}\} \quad \text{vectors} \quad \leftrightarrow \quad \sigma_{ij} \varepsilon_{ij} = \boldsymbol{\sigma}_i \boldsymbol{\varepsilon}_i$$

2<sup>nd</sup> order tensors

## VOIGT NOTATION

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$



$$\{\boldsymbol{\sigma}\} \stackrel{\text{def}}{=} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \in \mathbb{R}^3$$

# 2D Formulation in Voigt's Notation

- Hooke's law in terms of the stress and strain vectors:

$$\begin{cases} \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} \\ \mathbb{C} = \lambda \mathbf{1} \otimes \mathbf{1} + 2\mu \mathbb{I} \end{cases} \quad \rightarrow \quad \begin{cases} \{\boldsymbol{\sigma}\} = \mathbf{D} \cdot \{\boldsymbol{\varepsilon}\} \\ \sigma_i = D_{ij} \varepsilon_j \quad i \in \{1, 2, 3\} \end{cases}$$

Where  $\mathbf{D}$  is the 2-D matrix  
of elastic constants:

$$\mathbf{D} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

Plane stress

$$\mathbf{D} = \frac{\bar{E}}{1-\bar{\nu}^2} \begin{bmatrix} 1 & \bar{\nu} & 0 \\ \bar{\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\bar{\nu}}{2} \end{bmatrix}$$

Plane strain

being:

$$\bar{\nu} = \frac{\nu}{1-\nu}$$

$$\bar{E} = \frac{E}{1-\nu^2}$$



**END OF LECTURE 5**